QUARK MODEL PREDICTIONS FOR THE TRANSVERSITY AMPLITUDES[1]

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A simple derivation of the quark model relations between the transversity amplitudes is given. The problem of the choice of the spin reference frame in which the additivity assumption holds is discussed. It is shown that all these quark model relations can be obtained from some simple geometrical postulates.

1. Introduction

In spite of the fact that the additivity assumption in the quark model has been already used for a few years, even in the investigations of the polarization phenomena [1, 2] the simple relations between the reaction amplitudes (45), (62) have not been explicitly stated and investigated.

In this paper we derive in a very simple way the relations between the transversity amplitudes for the following processes:

$$P+B \to P + B^*, \tag{1}$$
$$P+B \to V + B^*, \tag{2}$$
$$B+B \to B^* + B^*, \tag{3}$$
$$B+B \to B + B^*, \tag{4}$$
$$B+B \to B + B, \tag{5}$$
$$P+B \to V + B, \tag{6}$$

where the following notation is used: $\boldsymbol{P}$ — pseudoscalar meson, $\boldsymbol{V}$ — vector meson, $\boldsymbol{B}$ — baryon from the $\frac{1}{2}^+$ octet, B^* — baryon from the $\frac{3}{2}^+$ decuplet.

The relations obtained are of three types $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ according to the classification given in [1]. The $\boldsymbol{a}$-type relations hold for the reactions (1–4) only, the $\boldsymbol{b}$- and $\boldsymbol{c}$-type relations hold for the reactions (2–6).

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The obtained relations hold also for the reactions obtained from the reactions (1–6) by putting in place of the particles — antiparticles.

It turns out that the $\boldsymbol{a}$-type relations can depend at most on one phenomenological parameter whereas the $\boldsymbol{b}$- and $\boldsymbol{c}$-type relations can depend on (at most) three and four phenomenological parameters. The discussion of the behaviour of the obtained relations under the rotation of the spin reference frame shows that the $\boldsymbol{c}$-type relations are not invariant under simultaneous rotation of all particles through the same angle.

It is shown that the $\boldsymbol{a}$-type relations can be obtained from the assumption that the amplitudes $f_{\mu_3\, {}^3\!/_2\, \mu_1 -{}^1\!/_2}$ and $f_{\mu_3 -{}^3\!/_2 \mu_1\, {}^1\!/_2}$ have to vanish in any spin reference frame obtained from some transversity frame by the simultaneous rotation of the spin quantization frames of the particles 2 and 4 through some β around the $\boldsymbol{y}_2$ and $\boldsymbol{y}_4$ axes respectively.

The $\boldsymbol{b}$- and $\boldsymbol{c}$-type relations are equivalent to the statement that the transversity amplitudes in some transversity frame are invariant under simultaneous rotations of the spin frames of all particles through the angle π around the $\boldsymbol{y}_i$ axes.

2. *The a-type relations*

Let us briefly recall the formalism of the additivity in the quark model. We consider the following types of reactions:

$$M_1+B_1 \to M_2+B_2, \tag{7}$$

$$M_1+B_1 \to M_2+B^* \tag{8}$$

and

$$B_1+B_2 \to \begin{cases} B_3+B_4 \\ B_3+B^* \end{cases}, \tag{9}$$

where $\boldsymbol{M}_i$ denotes a meson. The additivity assumption states how to calculate the transition amplitudes for the reactions (7) and (8) in terms of the quark-quark amplitudes. At first we write down the wave functions of quarks [3]. For the discussion of the additivity assumption it is convenient to adopt the following notation [4]:

$$\xi_1 = p_+, \quad \xi_2 = p_-, \quad \xi_3 = n_+,$$
$$\xi_4 = n_-, \quad \xi_5 = \lambda_+, \quad \xi_6 = \lambda_-,$$
$$\bar{\xi}_1 = \bar{p}_+, \ etc., \tag{10}$$

where $\boldsymbol{p}_+$ denotes the $\boldsymbol{p}$-quark wave function with the transversity value $\frac{1}{2}$ *etc.* Now we can write the wave functions of mesons and baryons in the following form:

$$M = \sum_{i,j} a(i,j)\xi_i\bar{\xi}_j,$$
$$B = \sum_{i,j,k} c(i,j,k)\xi_i\xi_j\xi_k,$$
$$B^* = \sum_{i,j,k} d(i,j,k)\xi_i\xi_j\xi_k \tag{11}$$

for $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} = 1, \dots 6$.

The transition amplitudes for the reactions (7) can be written in the form:

$$\langle M_2B_2|T|M_1B_1\rangle = \sum_{i,j,\ldots,w} a(i,j,l,m,n,q,r,s,t,w)\times$$
$$\times\langle \xi_i\bar{\xi}_j\xi_l\xi_m\xi_n|T|\xi_q\bar{\xi}_r\xi_s\xi_t\xi_w\rangle, \tag{12}$$

where $a(\boldsymbol{i},\boldsymbol{j},\boldsymbol{l},\boldsymbol{m},\boldsymbol{n},\boldsymbol{q},\boldsymbol{r},\boldsymbol{s},\boldsymbol{t},\boldsymbol{w})$ are known numbers. Now the additivity assumption states:

$$\langle \xi_i\bar{\xi}_j\xi_l\xi_m\xi_n|T|\xi_q\bar{\xi}_r\xi_s\xi_t\xi_w\rangle = \delta_{jr}\delta_{mt}\delta_{nw}\langle\xi_i\xi_l|\tau|\xi_q\xi_s\rangle +$$
$$+\delta_{jr}\delta_{ls}\delta_{nw}\langle\xi_i\xi_m|\tau|\xi_q\xi_t\rangle+\delta_{jr}\delta_{ls}\delta_{mt}\langle\xi_i\xi_n|\tau|\xi_q\xi_w\rangle+$$
$$+\delta_{iq}\delta_{mt}\delta_{nw}\langle\bar{\xi}_j\xi_l|\tau'|\bar{\xi}_r\xi_s\rangle+\delta_{iq}\delta_{ls}\delta_{nw}\langle\bar{\xi}_j\xi_m|\tau'|\bar{\xi}_r\xi_t\rangle+$$
$$+\delta_{iq}\delta_{mt}\delta_{ls}\langle\bar{\xi}_j\xi_n|\tau'|\bar{\xi}_r\xi_w\rangle$$
$$=\sum_{c,k}\left(\langle\xi_i\xi_c|\tau|\xi_q\xi_k\rangle\delta_{jr}+\langle\bar{\xi}_j\xi_c|\tau'|\bar{\xi}_r\xi_k\rangle\delta_{iq}\right)\delta_{b(c)b(k)}, \tag{13}$$

for $\boldsymbol{c}=\boldsymbol{m},\boldsymbol{l},\boldsymbol{n}$ and $\boldsymbol{k}=\boldsymbol{s},\boldsymbol{t},\boldsymbol{w}$, where $b(c)$ and $b(k)$ are three-dimensional vectors obtained from the index vectors $(\boldsymbol{l},\boldsymbol{m},\boldsymbol{n})$, $(\boldsymbol{s},\boldsymbol{t},\boldsymbol{w})$ in the following way:

$$b(l) = (o,m,n), \qquad b(m) = (l,o,n), \qquad b(n) = (l,m,o). \tag{14}$$

The vectors $b(\boldsymbol{k})$ are obtained in an analogous way from the index vector $(\boldsymbol{s},\boldsymbol{t},\boldsymbol{w})$. The symbol $\delta_{b(c)b(k)}$ has the meaning:

$$\delta_{b(c)b(k)} = \delta_{b_1(c)b_1(k)}\delta_{b_2(c)b_2(k)}\delta_{b_3(c)b_3(k)} \tag{15}$$

with

$$\delta_{00} = 1. \tag{16}$$

There is no formal change in the formula (12) when $\boldsymbol{B}_2$ is replaced by $\boldsymbol{B}^*$.

The amplitudes of the reactions (9) are evaluated with the help of the formula:

$$\langle B_3B_4|T|B_1B_2\rangle = \sum_{h,i,\ldots,w} b(h,i,j,l,m,n,p,q,r,s,t,w)\times$$
$$\times\langle\xi_h\xi_i\xi_j\xi_l\xi_m\xi_n|T|\xi_p\xi_q\xi_r\xi_s\xi_t\xi_w\rangle, \tag{17}$$

where

$$\langle\xi_h\xi_i\xi_j\xi_l\xi_m\xi_n|T|\xi_p\xi_q\xi_r\xi_s\xi_t\xi_w\rangle$$
$$=\sum_{c,d,e,k}\langle\xi_d\xi_c|\tau|\xi_e\xi_k\rangle\delta_{b(d)b(e)}\delta_{b(c)b(k)}. \tag{18}$$

The formula (18) is understood in the same way as the Eq. (13). As before there is no formal change in the Eq. (17) if the $\boldsymbol{B}_3$ and $\boldsymbol{B}_4$ are replaced by $\boldsymbol{B}_3^*$ or (and) $\boldsymbol{B}_4^*$.

To make all formulae shorter let us adopt the following notation:

$$\mathscr{S}\langle A_2\xi_l\xi_m\xi_n|T|A_1\xi_s\xi_t\xi_w\rangle$$
$$=\sum_{(l,m,n)}\sum_{(s,t,w)}\langle A_2\xi_l\xi_m\xi_n|T|A_1\xi_s\xi_t\xi_w\rangle \tag{19}$$

or

$$\mathscr{S}\langle \xi_i\xi_j\xi_k\xi_l\xi_m\xi_n|T|\xi_p\xi_q\xi_r\xi_s\xi_t\xi_w\rangle = \sum_{(\xi_l,\xi_m,\xi_n)}\sum_{(\xi_s,\xi_t,\xi_w)} \langle \xi_i\xi_j\xi_k\xi_l\xi_m\xi_n|T|\xi_p\xi_q\xi_r\xi_s\xi_t\xi_w\rangle \tag{20}$$

where the symbol $\sum_{(\xi_l,\,\xi_m,\,\xi_n)}$ means that the summation over all different permutations of the set $\xi_l\xi_m\xi_n$ should be performed.

From the Eqs (13), (18) and (19) it is easily seen that the following equalities hold:

$$\mathscr{S}\langle A_2\xi_l\xi_m\xi_n|T|A_1\xi_s\xi_t\xi_w\rangle = \begin{cases} \langle A_2\xi_l\xi_m\xi_n|T|A_1\xi_l\xi_m\xi_c\rangle\cdot 6 \\ \text{or } \langle A_2\xi_l\xi_m\xi_n|T|A_1\xi_l\xi_c\xi_n\rangle\cdot 6 \\ \text{or } \langle A_2\xi_l\xi_m\xi_n|T|A_1\xi_c\xi_m\xi_n\rangle\cdot 6 \end{cases} \tag{21}$$

and

$$\mathscr{S}\langle A_2\xi_l\xi_l\xi_m|T|A_1\xi_t\xi_s\xi_w\rangle = \begin{cases} \langle A_2\xi_l\xi_l\xi_m|T|A_1\xi_l\xi_l\xi_c\rangle\cdot 3, \\ \text{or } \langle A_2\xi_l\xi_l\xi_m|T|A_1\xi_l\xi_l\xi_m\rangle\cdot 6, \end{cases} \tag{22}$$

where $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ denote the meson or baryon, $b \neq l$, ξ_i denote the wave functions of the apropriate quarks and for ξ_c one can put any quarks not forbidden by conservation laws.

The wave functions of the baryons with the transversity value $\frac{1}{2}$ in terms of quarks can be written in the following form:

$$B\left(\frac{1}{2}\right) = \pm\frac{1}{\sqrt{18}}\left(2\sum_{(a,b)} a_+a_+b_- - \sum_{(a,b)} a_+a_-b_+\right) \tag{23}$$

where (a, b) is a shorthand notation of (a_+, a_+, b_-) or (a_+, a_-, b_+). The only exceptions are:

$$\Sigma^0 = \frac{1}{\sqrt{36}}\left(2\sum_{(p,n,\lambda)} p_+n_+\lambda_- - \sum_{(p,n,\lambda)} p_-n_+\lambda_+ - \sum_{(p,n,\lambda)} p_+n_-\lambda_+\right),$$

$$\Lambda^0 = \frac{1}{\sqrt{12}}\left(\sum_{(p,n,\lambda)} p_+n_-\lambda_+ - \sum_{(p,n,\lambda)} p_-n_+\lambda_+\right). \tag{24}$$

The $\frac{3}{2}$ baryon resonances wave functions with transversity value $\frac{3}{2}$ can be written in the following way:

$$B^*_{\mathrm{I}}\left(\frac{3}{2}\right) = a_+a_+a_+ \tag{25}$$

or

$$B^*_{\mathrm{II}}\left(\frac{3}{2}\right) = \frac{1}{\sqrt{3}}\sum_{(a,b)} a_+a_+b_+ \tag{26}$$

or

$$B^*_{\mathrm{III}}\left(\frac{3}{2}\right) = \frac{1}{\sqrt{6}}\sum_{(a,b,c)} a_+b_+c_+. \tag{27}$$

The functions with lower transversity values are obtained from those given above with the help of the well known spin lowering operator and the wave functions of the baryons and the baryon resonances $\boldsymbol{B}^*_{\mathrm{II}}$ are of the form:

$$B\left(-\frac{1}{2}\right) = \mp \frac{1}{\sqrt{18}}\left(2\sum_{(a,b)} a_-a_-b_+ - \sum_{(a,b)} a_+a_-b_-\right),$$

$$B^*_{\mathrm{II}}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\sum_{(a,b)} a_+a_+b_- + \sum_{(a,b)} a_+a_-b_+\right),$$

$$B^*_{\mathrm{II}}\left(-\frac{1}{2}\right) = \frac{1}{3}\left(\sum_{(a,b)} a_+a_-b_- + \sum_{(a,b)} a_-a_-b_+\right),$$

$$B^*_{\mathrm{II}}\left(-\frac{3}{2}\right) = \frac{1}{\sqrt{3}}\sum_{(a,b)} a_-a_-b_-. \tag{28}$$

For the baryon resonances of the type $\boldsymbol{B}^*_{\mathrm{I}}$ and $\boldsymbol{B}^*_{\mathrm{III}}$ the analogous formulae can be also obtained.

Now we shall calculate the following amplitudes:

$$\langle A_3(\mu_3)B^*_{\mathrm{II}}(\mu_4)|T|B_1(\mu_1)\, B_2(\mu_2)\rangle, \tag{29}$$

where A_3 denotes a baryon or baryon isobar, and

$$\langle M_2(\mu_3)B^*_{\mathrm{II}}(\mu_4)|T|M_1(\mu_1)B_2(\mu_2)\rangle. \tag{30}$$

If the baryon $\boldsymbol{B}_2$ is built from the aab quarks then the resonance $\boldsymbol{B}^*_{\mathrm{II}}$ can be built either from aab or aac or abb quarks. We shall calculate the amplitudes (29) for these three possibilities, however, the obtained relations for the amplitudes are independent of the quark contents of the resonance $\boldsymbol{B}^*_{\mathrm{II}}$.

With the help of Eq. (17) we have:

$$f_{\mu_3\mu_4\mu_1\mu_2} = \langle A_3(\mu_3)B^*_{\mathrm{II}}(\mu_4)|T|B_1(\mu_1)B_2(\mu_2)\rangle$$

$$= \sum_{(h,i,j)}\sum_{(p,q,r)} b_{\mu_3\mu_1}(h,i,j,p,q,r)\,\langle \xi_h\xi_i\xi_j B^*_{\mathrm{II}}(\mu_4)|T|\xi_p\xi_q\xi_r B_2(\mu_2)\rangle. \tag{31}$$

Now, using the Eqs (12–23) and (26) we obtain:

$$\left\langle \xi_h\xi_i\xi_j B^*_{\mathrm{II}}\left(\frac{3}{2}\right)\middle|T\middle|\,\xi_p\xi_q\xi_r B_2\left(\frac{1}{2}\right)\right\rangle = \pm\frac{1}{\sqrt{18}}\frac{1}{\sqrt{3}}\times$$

$$\times\left(2\mathscr{S}\,\langle\xi_h\xi_i\xi_j a_+a_+b_+|T|\xi_p\xi_q\xi_r a_+a_+b_-\rangle -\right.$$

$$\left.-\mathscr{S}\,\langle\xi_h\xi_i\xi_j a_+a_+b_+|T|\xi_p\xi_q\xi_r a_+a_-b_+\rangle\right)$$

$$= \pm\frac{1}{\sqrt{18}}\frac{6}{\sqrt{3}}\sum_{d,e}\left(\langle\xi_d b_+|\tau|\xi_e b_-\rangle - \langle\xi_d a_+|\tau|\xi_e a_-\rangle\right)\,\delta_{b(d)b(e)}. \tag{32}$$

For the baryon resonances with the quark contents aac and abb the amplitude (32) is equal:

$$\left\langle \xi_h\xi_i\xi_j B^*_{\mathrm{II}}\left(\frac{3}{2}\right)\middle|T\middle|\xi_p\xi_q\xi_r B_2\left(\frac{1}{2}\right)\right\rangle$$

$$= \pm\frac{1}{\sqrt{18}}\frac{6}{\sqrt{3}}\sum_{d,e}\langle\xi_d c_+|\tau|\xi_e b_-\rangle\,\delta_{b(d)b(e)} \tag{33}$$

and

$$\left\langle \xi_h\xi_i\xi_j B^*_{\mathrm{II}}\left(\frac{3}{2}\right)\middle|T\middle|\xi_p\xi_q\xi_r B_2\left(\frac{1}{2}\right)\right\rangle$$

$$= \mp\frac{1}{\sqrt{18}}\frac{6}{\sqrt{3}}\sum_{d,e}\xi_d b_+|\tau|\xi_e a_-\rangle\,\delta_{b(d)b(e)}. \tag{34}$$

In a similar way we obtain the amplitudes $F_i=\langle\xi_h\xi_i\xi_j B^*_{\mathrm{II}}(\tfrac{1}{2})|T|\xi_p\xi_q\xi_r B_2(-\tfrac{1}{2})\rangle$ for the possible quark contents of the resonance $\boldsymbol{B}^*_{\mathrm{II}}$:

$$F_1=\mp\frac{1}{\sqrt{18}}\frac{1}{3}\times(2\mathscr{S}\langle\xi_h\xi_i\xi_j a_+a_+b_-|T|\xi_p\xi_q\xi_r a_-a_-b_+\rangle+$$

$$+2\mathscr{S}\langle\xi_h\xi_i\xi_j a_+a_-b_+|T|\xi_p\xi_q\xi_r a_-a_-b_+\rangle-$$

$$-\mathscr{S}\langle\xi_h\xi_i\xi_j a_+a_+b_-|T|\xi_p\xi_q\xi_r a_+a_-b_-\rangle-$$

$$-\mathscr{S}\langle\xi_h\xi_i\xi_j a_+a_-b_+|T|\xi_p\xi_q\xi_r a_+a_-b_-\rangle)$$

$$=\mp\frac{1}{\sqrt{18}}\frac{6}{3}\sum_{d,e}(\langle\xi_d a_+|\tau|\xi_e a_-\rangle-\langle\xi_d b_+|\tau|\xi_e b_-\rangle)\,\delta_{b(d)b(e)} \tag{35}$$

for the quark content aab and

$$F_2=\pm\frac{1}{\sqrt{18}}\frac{6}{3}\sum_{d,e}\langle\xi_d c_+|\tau|\xi_e b_-\rangle\,\delta_{b(d)b(e)} \tag{36}$$

for the quark content aac or

$$F_3=\mp\frac{1}{\sqrt{18}}\frac{6}{3}\sum_{d,e}\langle\xi_d b_+|\tau|\xi_e a_-\rangle\,\delta_{b(d)b(e)} \tag{37}$$

for the quark content abb.

Now comparing suitable equations we obtain the first relations between the amplitudes:

$$\frac{1}{\sqrt{3}}f_{\mu_3\,{}^3/_2\,\mu_1\,{}^1/_2}=f_{\mu_3\,{}^1/_2\,\mu_1\,-{}^1/_2}. \tag{38}$$

In the same way the other relations between the amplitudes of that reaction are obtained:

$$f_{\mu_3\,{}^1/_2\,\mu_1\,{}^1/_2}=f_{\mu_3\,-{}^1/_2\,\mu_1\,-{}^1/_2} \tag{39}$$

and

$$\frac{1}{\sqrt{3}}f_{\mu_3\,-{}^3/_2\,\mu_1\,-{}^1/_2}=f_{\mu_3\,-{}^1/_2\,\mu_1\,{}^1/_2}. \tag{40}$$

It is also easy to show that

$$f_{\mu_3\mu_4\mu_1\mu_2} = 0 \tag{41}$$

for $|\mu_4 - \mu_2| > 1$.

Though the relations (38–41) are proved only for reactions of the type:

$$B_1 + B_2 \to A_3 + B^*_{II}, \tag{42}$$

where $\boldsymbol{A}_3$ denotes $\boldsymbol{B}$ or $\boldsymbol{B}^*$, it can easily be seen that they also hold if $\boldsymbol{B}^*_{II}$ is replaced by $\boldsymbol{B}^*_{I}$ or $\boldsymbol{B}^*_{III}$. Equally easy is generalization on the reactions (7–8) with the help of Eq. (10) and (13), one obtains at once the amplitudes for the reaction:

$$M_1 + B \to M_2 + B^*_{II} \tag{43}$$

from the amplitudes for the reaction (42). For example from (31) and (32) we have:

$$f_{\mu_3\,^3/_2\,\mu_1\,^1/_2} = \left\langle M_2(\mu_3) B^*_{II}\left(\frac{3}{2}\right)\right|T\left|M_1(\mu_1) B_2(\mu_2)\right\rangle$$

$$= \pm \frac{1}{\sqrt{18}} \frac{6}{\sqrt{3}} \sum_{i,j,q,r} a_{\mu_3\mu_1}(i,j,q,r)\, [(\langle \xi_i b_+ | \tau | \xi_q b_- \rangle - \langle \xi_i a_+ | \tau | \xi_q a_- \rangle)\, \delta_{jr} +$$

$$+ (\langle \bar{\xi}_j b_+ | \tau' | \bar{\xi}_r b_- \rangle - \langle \bar{\xi}_j a_+ | \tau' | \bar{\xi}_r a_- \rangle)\, \delta_{iq}\,. \tag{44}$$

It is obvious that to obtain (44) we do not have to repeat all the calculations made in (32), it is also seen that all the relations (38–41) hold automatically for the reaction (43). If we replace $\boldsymbol{B}^*_{II}$ in (43) by $\boldsymbol{B}^*_{I}$ or $\boldsymbol{B}^*_{III}$ the relations (38–41) will be also valid.

The relations (41) also stem from a weaker condition than the additivity assumption, that the value of the transversity can be changed only by 1, 0 or -1.

Let us notice that the relations (38–41) can be conveniently written in the form:

$$f_{\mu_3\mu_4\mu_1\mu_2} = N_{3/2}(\mu_4) N_{1/2}(\mu_2) f_{\mu_3\mu_4+1\,\mu_1\mu_2+1} + \\ + N_{3/2}(-\mu_4) N_{1/2}(-\mu_2) f_{\mu_3\mu_4-1\,\mu_1\mu_2-1} \tag{45}$$

with $N_{1/2}$ and $N_{3/2}$ coefficients defined as follows:

$$N_{1/2}(\mu_2) = \begin{cases} 1 \text{ for } \mu_2 = -\tfrac{1}{2}, \\ 0 \text{ for all other cases} \end{cases} \tag{46}$$

and

$$N_{3/2}(\mu_4) = \begin{cases} \frac{1}{\sqrt{3}} & \text{for } \mu_4 = \frac{1}{2}, \\ 1 & \text{for } \mu_4 = -\frac{1}{2}, \\ \sqrt{3} & \text{for } \mu_4 = -\frac{3}{2}, \\ 0 & \text{for all other cases.} \end{cases} \tag{47}$$

It also immediately follows from our considerations that the full set of relations for the reaction:

$$B_1+B_2 \to B_1^*+B_2^* \tag{48}$$

consists of (45) and of relations:

$$\begin{aligned} f_{\mu_3\mu_4\mu_1\mu_2} &= N_{3/2}(\mu_3)N_{1/2}(\mu_1)\, f_{\mu_3+1\,\mu_2\mu_1+1\,\mu_4}+ \\ &+N_{3/2}(-\mu_3)N_{1/2}(-\mu_1)\, f_{\mu_3-1\,\mu_2\mu_1-1\,\mu_4}. \end{aligned} \tag{49}$$

From the relations (45) and (49) one can obtain another relation:

$$\begin{aligned} f_{\mu_3\mu_4\mu_1\mu_2} &= N_{3/2}(\mu_3)N_{3/2}(\mu_4)N_{1/2}(\mu_1)N_{1/2}(\mu_2)\, f_{\mu_3+1\,\mu_4+1\,\mu_1+1\,\mu_2+1}+ \\ &+N_{3/2}(\mu_3)N_{3/2}(-\mu_4)N_{1/2}(\mu_1)N_{1/2}(-\mu_2)\, f_{\mu_3+1\,\mu_4-1\,\mu_1+1\,\mu_2-1}+ \\ &+N_{3/2}(-\mu_3)N_{3/2}(\mu_4)N_{1/2}(-\mu_1)N_{1/2}(\mu_2)\, f_{\mu_3-1\,\mu_4+1\,\mu_1-1\,\mu_2+1}+ \\ &+N_{3/2}(-\mu_3)N_{3/2}(-\mu_4)N_{1/2}(-\mu_1)N_{1/2}(-\mu_2)\, f_{\mu_3-1\,\mu_4-1\,\mu_1-1\,\mu_2-1}. \end{aligned} \tag{50}$$

We should like to point out here that till now all the obtained relations were independent of the detailed properties of the quark-quark amplitudes; they are so called **a**-type relations [1].

3. *The* **b**- *and* **c**-*type relations*

We obtain the so called **b**-type relations if we assume that the following relations between quark-quark amplitudes hold:

$$\langle a_+b_-|\tau|c_-d_+\rangle = \langle a_-b_+|\tau|c_+d_-\rangle. \tag{51}$$

The relations (51) and the equality:

$$\langle a_-b_-|\tau|c_+d_+\rangle = \langle a_+b_+|\tau|c_-d_-\rangle, \tag{52}$$

where a, b, c, d denote arbitrary quarks or antiquarks, yield the **c**-type relations [1].

To obtain the relations (2–6) let us introduce the linear operator $\hat{\boldsymbol{I}} = \hat{\boldsymbol{I}}_1+\hat{\boldsymbol{I}}_2+\hat{\boldsymbol{I}}_3$ where $\hat{\boldsymbol{I}}_i$ is an operator reversing the signs of the transversity value of the quarks standing on the $\boldsymbol{i}$-th place for example $\hat{\boldsymbol{I}}\, a_+a_+b_- = a_-a_-b_+$. The $\hat{\boldsymbol{I}}$ acts in the following way on the baryon, meson and baryon isobar wave functions:

$$\hat{I}B(\mu) = -B(-\mu), \tag{53}$$

$$\hat{I}B^*_{\mathrm{I,\,II,\,III}}(\mu) = B^*_{\mathrm{I,\,II,\,III}}(-\mu), \tag{54}$$

$$\hat{I}V(\mu) = V(-\mu), \tag{55}$$

$$\hat{I}P = -P. \tag{56}$$

By the use of $\hat{\boldsymbol{I}}$ we can rewrite the relations (51) and (52) in the following way:

$$\begin{aligned} \langle \xi_i\xi_j|\tau|\xi_k\xi_l\rangle &= \langle \hat{I}\xi_i\hat{I}\xi_j|\tau|\hat{I}\xi_k\hat{I}\xi_l\rangle, \\ \langle \xi_i\bar{\xi}_j|\tau'|\xi_k\bar{\xi}_l\rangle &= \langle \hat{I}\xi_i\hat{I}\bar{\xi}_j|\tau'|\hat{I}\xi_k\hat{I}\bar{\xi}_l\rangle. \end{aligned} \tag{57}$$

From the additivity assumption (13) it follows that each amplitude for the reactions (1–6) is a linear combination of the quark-quark and quark-antiquark amplitudes

$$f_{\mu_3\mu_4\mu_1\mu_2} = \langle A_3(\mu_3)A_4(\mu_4)|T|A_1(\mu_1)A_2(\mu_2)\rangle$$
$$= \sum_{i,j,k,l} \alpha_{i,j,k,l}(\mu_3, \mu_4, \mu_1, \mu_2) \langle \xi_i\xi_j|\tau|\xi_k\xi_l\rangle +$$
$$+ \sum_{i,j,k,l} \beta_{i,j,k,l}(\mu_3, \mu_4, \mu_1, \mu_2) \langle \xi_i\bar{\xi}_j|\tau'|\xi_k\bar{\xi}_l\rangle. \quad (58)$$

Now let us see that:

$$\hat{I}f_{\mu_3\mu_4\mu_1\mu_2} = \langle \hat{I}A_3(\mu_3)\hat{I}A_4(\mu_4)|T|\hat{I}A_1(\mu_1)\hat{I}A_2(\mu_2)\rangle$$
$$= \sum_{i,j,k,l} \alpha_{i,j,k,l}(\mu_3, \mu_4, \mu_1, \mu_2) \langle \hat{I}\xi_i\hat{I}\xi_j|\tau|\hat{I}\xi_k\hat{I}\xi_l\rangle +$$
$$+ \sum_{i,j,k,l} \beta_{i,j,k,l}(\mu_3, \mu_4, \mu_1, \mu_2) \langle \hat{I}\xi_i\hat{I}\bar{\xi}_j|\tau'|\hat{I}\xi_k\hat{I}\bar{\xi}_l\rangle. \quad (59)$$

From the Eq. (51), (52), (57–59) we immediately have

$$\hat{I}f_{\mu_3\mu_4\mu_1\mu_2} = f_{\mu_3\mu_4\mu_1\mu_2} \quad (60)$$

for $\mu_1 - \mu_3 = \mu_4 - \mu_2 \neq 0$ for the b-type relations and for $|\mu_1 - \mu_3| = |\mu_4 - \mu_2| \neq 0$ for the c-type relations. Now from the Eq. (53–56) and (59) we come to the relations

$$\hat{I}f_{\mu_3\mu_4\mu_1\mu_2} = (-1)^N f_{-\mu_3-\mu_4-\mu_1-\mu_2}, \quad (61)$$

where N denotes the total number of baryon and pseudoscalar mesons participating in the reaction. The Eq. (60) and (61) give us required relations between reaction amplitudes in the following form:

$$f_{\mu_3\mu_4\mu_1\mu_2} = (-1)^N f_{-\mu_3-\mu_4-\mu_1-\mu_2}, \quad (62)$$

or $\mu_1 - \mu_3 = \mu_4 - \mu_2 \neq 0$ in the case of b-type relations and for $|\mu_1 - \mu_3| = |\mu_4 - \mu_2| \neq 0$ n t he case of c-type relations.

4. *Phenomenological parameters*

In the preceding sections we have found the relations between the amplitudes in the transversity frame in which the additivity assumption in the quark model holds (in the following we will call this frame the additivity frame). But it is well known that the transversity frames are those in which the normal to the reaction plane was chosen as the spin quantization axis for all the particles participating in the reaction [5]. Therefore, we have a lot of transversity frames which can be obtained from each other by the rotation of the spin reference frame of the i-th particle through some angle ψ_i around the normal to the reaction plane. All the relations (45), (49), (50) and (62) do not depend on the particular choice of the additivity frame. However, if we want to check experimentally the quark model relations in some fixed transversity frame for example in the standard one in which the x_i-axis of the i-th particle is in the direction of its linear momentum in the overall center of mass frame, we have to find how these relations look like in this frame. Because we do not know which of the transversity frames is the additivity frame so we have to introduce the phenomenological parameters φ_i which have the meaning of angles through which we have to rotate the standard transversity frame to obtain the additivity frame. With the help of [6] we obtain the following equalities:

$$f^{A}_{\mu_3\mu_4\mu_1\mu_2} = e^{i(\mu_3\varphi_3+\mu_4\varphi_4-\mu_1\varphi_1-\mu_2\varphi_2)} f_{\mu_3\mu_4\mu_1\mu_2}, \tag{63}$$

where $f^{A}_{\mu_3\mu_4\mu_1\mu_2}$ denote the amplitudes in the additivity frame and $f_{\mu_3\mu_4\mu_1\mu_2}$ in the standard one. Now using the Eq. (63) we obtain from (49) and (62) the following relations between the amplitudes in the standard transversity frame:

$$f_{\mu_3\mu_4\mu_1\mu_2} = e^{i(\varphi_4-\varphi_2)} N_{3/2}(\mu_4) N_{1/2}(\mu_2) f_{\mu_3\mu_4+1\,\mu_1\mu_2+1} + \\ + e^{-i(\varphi_4-\varphi_2)} N_{3/2}(-\mu_4) N_{1/2}(-\mu_2) f_{\mu_3\mu_4-1\,\mu_1\mu_2-1}, \tag{64}$$

$$f_{\mu_3\mu_4\mu_1\mu_2} = (-1)^N e^{-2i[\mu_1(\varphi_4-\varphi_2)+\mu_2(\varphi_4-\varphi_1)-\mu_3(\varphi_4-\varphi_3)]} \times \\ \times f_{-\mu_3-\mu_4-\mu_1-\mu_2} \tag{65}$$

for $\mu_3-\mu_1 = \mu_2-\mu_4 \neq 0$,

$$f_{\mu_3\mu_4\mu_1\mu_2} = (-1)^N e^{-2i[\mu_1(\varphi_4-\varphi_2)-\mu_2(\varphi_4+\varphi_1)+\mu_3(\varphi_4+\varphi_3)]} \times \\ \times f_{-\mu_3-\mu_4-\mu_1-\mu_2} \tag{66}$$

or $\mu_3-\mu_1 = \mu_4-\mu_2 \neq 0$.

As we see the **a**-type relations (64) depend at most on one parameter and they are invariant under the rotations of the spin reference frame for which $\psi_4 = \psi_2$ but ψ_1 and ψ_3 can be arbitrary.

The **b**-type relations (65) depend at most on three parameters and they are invariant funder rotations for which $\psi_1 = \psi_2 = \psi_3 = \psi_4$.

The **c**-type relations (65) and (66) depend at most on four parameters, however, unless $\mu_3 = \mu_2$, they are not invariant under the rotations in which $\psi_1 = \psi_2 = \psi_3 = \psi_4$. They are only invariant under the rotations for which $\psi_1 = \psi_2 = \psi_3 = -\psi_4$.

The peculiar behaviour of the **c**-type relations under the simultaneous rotation of the spin quantization frames for each particle through the same angle which corresponds to the simultaneous rotation of the spins of all interacting particles through the same angle seems to suggest that these relations are not correct. It was shown that the **c**-type predictions for the statistical tensors are consistent with the experimental data [2]. However, the method used involved fitting of the averaged $q-q$ amplitudes, so it does not seem to be certain. Recently Kielanowski [7] found the **c**-type predictions for the generalized statistical tensors [8] for the reaction (2). To check them one need not to fit any quark-quark amplitudes, so we think that checking these relations one can obtain more reliable answer whether the **c**-type relations are in agreement with the experimental data.

At the end of this section we want to stress that the parameters φ_i can depend on E, ϑ and on the type of the process.

5. *The geometrical nature of the quark model relations*

We will discuss in this section a very interesting implication of the quark model, namely that all the quark model predictions for the amplitudes can be obtained from some geometrical assumptions. The derivation is very simple and immediate but the physical meaning of these assumptions is not completely clear.

Let us assume that in the transversity frame

$$f_{\mu_3\mu_4\mu_1\mu_2} = 0 \text{ for } |\mu_4-\mu_2| > 1 \text{ (or } |\mu_3-\mu_1| > 1).$$

Our statement is as follows:

There exists one particular transversity frame $\boldsymbol{A}$ such that the amplitude:

$$f'_{\mu_3\mu_4\mu_1\mu_2}(\beta) = 0 \text{ for } |\mu_4-\mu_2| > 1 \text{ (or } |\mu_3-\mu_1| > 1) \tag{67}$$

in all the frames obtained from the frame $\boldsymbol{A}$ by the simultaneous rotation of the spin frames of the particles *2* and *4* (or *1* and *3*) through the same angle β around the $\boldsymbol{y}_2$ and $\boldsymbol{y}_4$ (or $\boldsymbol{y}_1$ and $\boldsymbol{y}_3$) axes respectively.

We will show that this statement leads to relations between the amplitudes in the frame $\boldsymbol{A}$ which are exactly the same as the $\boldsymbol{a}$-type relations in the additivity frame (38–41). However, in this method it is not clear why such a frame $\boldsymbol{A}$ should exist.

For the reactions a $\boldsymbol{B}-\boldsymbol{B}^*$ vertex the statement (67) has the form:

$$f'_{\mu_3\,{}^3/_2\,\mu_1\,-{}^1/_2}(\beta) = 0, \tag{68}$$

$$f'_{\mu_3\,-{}^3/_2\,\mu_1\,{}^1/_2}(\beta) = 0. \tag{69}$$

for an arbitrary β.

Now using the transformation properties of the transversity amplitudes under the rotation and the explicit form of the $d^j_{mm'}(\beta)$ functions [6] we obtain from (68):

$$0 = f'_{\mu_3\,{}^3/_2\,\mu_1-{}^1/_2}(\beta) = \sum_{\mu_4',\mu_2'} d^{3/2}_{\mu_4'\,{}^3/_2}(\beta)d^{1/2}_{\mu_2'-{}^1/_2}(\beta)f_{\mu_3\mu_4'\mu_1\mu_2'}$$

$$= -\frac{1+\cos\beta}{4}\sin\beta\,(f_{\mu_3\,{}^3/_2\,\mu_1\,{}^1/_2} - \sqrt{3}f_{\mu_3\,{}^1/_2\,\mu_1-{}^1/_2}) -$$

$$-\sqrt{3}\,\frac{1+\cos\beta}{2}\sin^2\frac{\beta}{2}\,(f_{\mu_3\,{}^1/_2\,\mu_1\,{}^1/_2} - f_{\mu_3-{}^1/_2\,\mu_1-{}^1/_2}) -$$

$$-\frac{1-\cos\beta}{4}\sin\beta\,(\sqrt{3}f_{\mu_3-{}^1/_2\,\mu_1\,{}^1/_2} - f_{\mu_3-{}^3/_2\,\mu_1-{}^1/_2}). \tag{70}$$

All other terms vanish. Since this equality has to hold for an arbitrary β, all expressions in the brackets have to vanish, so we obtain the relations (38–40). The Eq. (69) leads to the same relations.

The $\boldsymbol{b}$- and $\boldsymbol{c}$-type relations are equivalent to another geometrical postulate that the amplitude:

$$f''_{\mu_3\mu_4\mu_1\mu_2} = f_{\mu_3\mu_4\mu_1\mu_2} \tag{71}$$

for $\mu_1-\mu_3=\mu_4-\mu_2\neq 0$ or for $|\mu_1-\mu_3|=|\mu_4-\mu_2|\neq 0$, where $f_{\mu_3\mu_4\mu_1\mu_2}$ is the reaction amplitude in the frame $\boldsymbol{A}$ and $f''_{\mu_3\mu_4\mu_1\mu_2}$ is the reaction amplitude in the frame obtained from the frame $\boldsymbol{A}$ by the simultaneous rotation of the spin frames of all the particles through the angle π around the $\boldsymbol{y}_i$-axes. The proof is simple.

6. *Discussion*

At first we should point out that we obtain the same relations between amplitudes for the reactions of type (1–6) with all particles replaced by antiparticles. It is also seen that if we want to investigate the processes of the types:

$$\gamma+B \rightarrow V+B, \tag{72}$$

$$\gamma+B \rightarrow V+B^* \tag{73}$$

where γ is photon, in terms of the quark model, we can obtain in a similar way the $\boldsymbol{a}$-, $\boldsymbol{b}$-, $\boldsymbol{c}$-type relations for the reaction (72) and the $\boldsymbol{b}$-, $\boldsymbol{c}$-type relations for the reaction (71).

We should like to stress that usually the quark model predictions for measurable quantities which are functions of the reaction amplitudes have been obtained after quite lengthy calculations. The method was based on expressing the measurable quantities by the quark-quark amplitudes and then on looking for relations between the former. Sometimes it led to separate calculations for the 70 existing reactions. It was very hard to say which of the quark model assumptions were necessary to obtain the final relations between measurable quantities. Sometimes checking of the obtained relations involved fitting of the quark--quark amplitudes [2].

Having simple relations between the amplitudes one has a much simpler way of deriving the predictions for the measurable quantities and the possibility of the discussion of the problem of the necessity of the adopted assumptions.

In the next papers we shall derive the relations between the measurable quantities — the generalized statistical tensors.

At the end we want to underline that the positive experimental check of the $\boldsymbol{a}$-, $\boldsymbol{b}$-$\boldsymbol{c}$-type relations between the reaction amplitudes cannot be treated as a good check of the additivity assumption in the quark model because only some of the quark model assumptions were used. However, it can be treated as a good check of the geometrical postulates (67) and (70).

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The note added in proof;

There is a lot of the additivity assumptions in the quark model. The one used above was based on the non-relativistic, additivity assumption. However, our reasoning can be applied to any additivity assumption, which, in some spin quantization reference frame, gives the decomposition of the reaction amplitudes of the type (12) and (13). For the discussion of the different additivity assumptions the reader is referred to [9]. Also in [9] it is shown how from the different additivity assumptions using the results of this paper one can obtain the same relations for the transversity amplitudes.