Tests for the Purity of the Initial Ensemble of States in Scattering Experiments.

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Recently a hypothesis was formulated [1] that the initial ensemble of two-particle states E is a mixed quantum ensemble with respect to the set of quantum numbers $\alpha = \{S, \Lambda, l, \sigma\}$, where S denotes the total invariant spin, Λ its projection on the direction of the total linear momentum, l the relative orbital momentum and σ the resultant spin of the particles. In that case [1] an initial two-particle state is represented by the density matrix $\hat{D}^i$

$$\hat{D}^i = \sum_{\alpha} |\alpha, \beta, \gamma\rangle \varrho(\alpha) \langle \alpha, \beta, \gamma| . \tag{1}$$

The need for systematic tests (P.T.) for the purity of the initial ensemble E has been indicated.

In this paper we propose such tests and we discuss them referring to formula (1), however they are also applicable if the ensemble E is a mixture of different type [1-3].

The idea of P.T. is very simple. One has to find the dependence of the estimated values of some physical quantities characterizing the scattering (weakly dependent on spin) on the geometry of the beam-target configuration.

But experimentalists could say that the lack of such dependence is carefully checked in all good experiments, so the problem is solved.

However, revealing the mixed character of initial states of type (1) can be quite difficult. Different mixed states of this type lead to different probabilities of finding the beam and the target particles in the range of their strong mutual interaction. One cannot control the detailed momentum distribution of the beam and target particles and hence also the distribution of their relative momenta with an accuracy sufficient to fix the above-mentioned probabilities. Thus it is very probable that $\varrho(\alpha)$ in formula (1) is a very rapidly oscillating function of time $\varrho(\alpha) = \varrho(\alpha, t)$. If the data for

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(*) We recall that we are mainly interested in finding the mixed character of $\hat{D}^i$ with respect to 1.

(1) M. KUPCZYŃSKI: *Lett. Nuovo Cimento*, **9**, 134 (1974); **11**, 117 (1974).
(2) M. KUPCZYŃSKI: *Phys. Lett.*, **47** B, 244 (1973).
(3) P. H. EBERHARD: *Should unitarity be tested experimentally*, CERN 72-1.

P.T. were gathered in time intervals much longer than the period of the above-mentioned oscillations, all the structure could be averaged out and the result of P.T. misleading. Before our propositions on the P.T. we state the problem in the language of mathematical statistics.

If the initial state is pure and in the quantum-mechanical theoretical description is represented by the vector $|\alpha\rangle$, then as an outcome of the experiment one obtains a sample drawn from some homogeneous statistical population P_α of the observed values of some physical quantity Q_α. The distribution $F(Q_\alpha)$ of the population is unknown; however in most physical applications it is assumed that an appropriate variant of the central limit theorem holds, so the arithmetic means of the n-(large) arbitrary observed values of Q_α should be normally distributed in the limit $n \to \infty$. Thus one can consider samples drawn from an appropriate normal population $\bar{P}_\alpha$ having a normal distribution $N(Q_\alpha)$. The different statistical methods enable one to estimate the mean and the variance of the distribution $N(Q_\alpha)$ and consequently the mean and the variance of $F(Q_\alpha)$.

For the mixed states (1) one deals with a particular mixture of the samples drawn from the different homogeneous populations P_α (or put differently with a sample drawn from a mixed population P_m). To check the purity of the initial states one has to check the homogeneity of the samples from the population P_m or of the corresponding normal population $\bar{P}_m$. A different check can be carried out by considering one- or two-parameter families of samples $S(\lambda, \gamma)$, where the parameters λ and γ correspond to the different arrangements of the experiment or different gathering or grouping of the data, which could change the mixed population P_m into the different mixed population P'_m. The P.T. reduces in that case to checking whether, for different λ and γ, $S(\lambda, \gamma)$ are the different samples drawn from the same unknown general population P or whether they are the samples drawn from the different populations $P(\lambda, \gamma)$.

Different statistical tests can be useful for this purpose, for example variants of tests *a*) and *d*) from Subsect. **11**.3.2 of ([4]), the tests based on the Kolmogorov-Smirnov theorems ([5]), or the ones based on the series analysis ([6]) and perhaps many others appropriately adapted from the tests presented in ([7-9]). A review of the statistical tests appropriate for the purity testing will be given in another paper.

Before describing our porposals for experiments we make a general remark about the quantity Q which should be used in the P.T. The evaluation of the different cross-sections from the data is quite cumbersome and indirect (see for example ([10])). Thus *the best quantities* Q *for the* P.T. *are, in our opinion, different flux (and, if possible, spin) independent ratios* $I_1(\lambda, \gamma)/I_m(\lambda, \gamma)$ *of the counting rates* $I_1(\lambda, \gamma)$ *and* $I_m(\lambda, \gamma)$ *obtained by procedures indexed by* λ *and* γ.

Our proposals on experiments and the quantities Q are the following:

1) The usual « good-geometry » transmission experiment used for measuring total cross-sections. We describe a scheme from ([10]). A well-focused beam from the accelerator (with a small spread of linear momenta) is cast on the hydrogen target. In front of the target one has monitor counters C_M and behind the target one has three detectors C_5, C_6, C_7 of very small size placed on a line at some distances d_i each from the other, working in coincidence with the monitor counters C_M and registering those

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([4]) B. A. MARTIN: *Statistics for Physicists* (London and New York, 1971).
([5]) J. L. DOOB: *Ann. Math. Stat.*, **20**, 393 (1949).
([6]) A. WALD and J. WOLFOWITZ: *Ann. Math. Stat.*, **11**, 147 (1949).
([7]) A. WALD: *Statistic Decision Functions* (New York, N. Y., 1950).
([8]) L. JANOSSY: *Theory and Practice of the Evaluation of the Measurements* (Oxford, 1965).
([9]) P. KENDALL and A. STUART: *The Advanced Theory of Statistics* (1958).
([10]) U. AMALDI, T. FAZZINI, G. FIDECARO, C. GHESQUIÈRE, M. LEGROS and H. STEINER: *Nuovo Cimento*, **34**, 825 (1964).

particles from the beam which were not scattered by the target. One can change the experimental position of the focus of the beam f, the transverse position of the beam image relative to the centre of the target, the distances d_i and the periods of time Δt used for the evaluation of the counting rates for all the counters. If the set of those parameters and the rates measured by the counters C_M, C_5, C_6, C_7 are denoted by λ, I_M, I_5, I_6 and I_7, respectively, then for the quantity Q we can take $I_i(\lambda)/I_M(\lambda)$, $I_7(\lambda)/(I_6(\lambda)-I_7(\lambda))$, $(I_5(\lambda)-I_6(\lambda))/I_7(\lambda)$ (only the intensities $I_7(\lambda)$, $I_6(\lambda)-I_7(\lambda)$, $I_5(\lambda)-I_6(\lambda)$ are independent ([10])). Thus we obtain the samples $S(\lambda)$ which we analyse by the above-mentioned methods. The strongest effect should be presumably obtained for small Δt (to prevent averaging) and for the different values of the parameter λ. Due to the possibility of admixtures of multiple scattering, the interpretation of the results of this test can be a little ambiguous.

2) The ISR experiments provide us with clean tests for purity. Multiple-scattering effects are eliminated. In the ISR (ref. ([11]) and the references therein) one can displace the two beams vertically in small and known steps δ (changing in this way the geometry of the intersecting region and possibly the mixed initial state). Due to the Van der Meer method ([12]), if the initial ensemble of states is pure, the cross-section $\Delta\sigma_M$ of the events which are within the acceptance of the monitor system is connected with the monitor rate $R_M(\delta)$, being a function of the displacement δ by the following formula:

$$R_M(\delta) = \frac{\Delta\sigma_M}{e^2 c \operatorname{tg}(\gamma/2)} \int_{-\infty}^{\infty} i_1(z)\, i_2(z+\delta)\, \mathrm{d}z\,, \tag{2}$$

where $\Delta\sigma_M$ should not depend on δ.

Thus if we observe the same beam intersection simultaneously by the two monitor systems M_1 and M_2, then

$$\frac{R_{M_1}(\delta, \Delta t)}{R_{M_2}(\delta, \Delta t)} = \frac{\Delta\sigma_{M_1}}{\Delta\sigma_{M_2}} = \text{const}\,, \tag{3}$$

where Δt is a period in whic counts are gathered for the evaluation of the rates. *Only if the initial ensemble is mixed* $Q = R_{M_1}(\delta, \Delta t)/R_{M_2}(\delta, \Delta t)$ *may be* Δt *and* δ *dependent.* Changing M_1, M_2, δ, Δt one obtains a huge number of P.T. If one has an on line computer registering the events as in the experiment ([13]), one can even investigate the order of the event statistics and search for its dependence on δ or Δt.

3) The analysis of pictures from the hydrogen bubble chambers can also provide P.T., however the time order of the expositions has to be kept. For example, one can consider multiplicity distributions. Namely one can investigate the distributions of the quantity Q

$$Q = (I_n/I_m)\,\Delta t\,,$$

([11]) U. AMALDI: NP Internal Report 73-5; C. BELLETINI: talk at *Frascati Meeting* (1973), both to be published in *Highligts in Particle Physics*, edited by A. ZICHICHI (Bologna).

([12]) J. VAN DER MEER: CERN Internal Report ISR-Po/68-31 (1968), unpublished.

([13]) U. AMALDI, R. BIANCASTELLI, C. BOSIO, G. MATTHIAE, J. V. ALLABY, W. BARTEL, M. M. BLOCK, G. COCCONI, A. N. DIDDENS, R. W. DOBINSON, J. LITT and A. M. WETHERELL: *Phys. Lett.*, **43** B, 231 (1973).

where I_n and I_m are observed numbers of the n- and m-prong events, respectively, in the periods of time Δt. Changing reasonably Δt one can check the dependence of the distributions of Q on Δt.

Any result of P.T. is very interesting. If the purity of the initial states were confirmed, then we would be sure that the use of the conventional description is legitimate. If the initial beam would turn out to be mixed, it would be an exciting discovery with far reaching consequences. We hope that the experimentalists will discover better tests than those presented in this paper.

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