BERTRAND'S PARADOX AND BELL'S INEQUALITIES

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Bertrand's paradox from the theory of probability is recalled and its relevance to the impossibility of proving Bell's inequalities for realistic spin polarization experiments is pointed out.

The results of spin polarization correlation experiments (SPCE) together with Bell's inequalities are sometimes considered as a proof that any theory with supplementary parameters (TSP) able to reproduce the quantum mechanical predictions (QMP) has to violate einsteinian causality [1,2].

On the other hand local deterministic spin models reproducing QMP have been constructed [3,4] and random experiments with macroscopic bodies violating the Bell inequalities have been invented and performed [5].

There is a lot of confusion [1] concerning these contradictory results; this is why we want to reexamine the Bell inequalities. The main hypothesis (H) needed to prove Bell-type inequalities [6–9] is the assumption that the probabilities estimated in various SPCE can be calulated from *one* sample space (probability space) by conditionalization. The hypothesis H is not valid in the refs. [3–5], thus the Bell inequalities do not apply. Moreover we are going to show that H implies serious constraints on the way in which random experiments should be performed, which turned out to be not satisfied in SPCE.

Before elaborating this point we want to recall the instructive paradox contemplated for the first time by Bertrand [10].

Let us consider two concentrical circles on a plane with radii R and $R/2$, respectively. If we ask the question: "What is the probability P that a chord of the bigger circle chosen at random cuts the smaller one at least at one point? ", we may find, using simple geometrical arguments, three evident contradictory values of P. Namely, if we divide the ensemble of chords into the subensembles of parallel chords, we find $P=1/2$; if we consider the subensembles of chords having the same beginning, we find $P=1/3$ and if we look for the midpoints of the chords lying in the small circle, we find $P=1/4$. The explication of these seemingly paradoxical results is well known [11]. The different values of P correspond to the different random experiments which may be used to find the answer to the Bertrand question. Thus the probabilities have only a precise meaning if the random experiments used for their estimation are specified.

Let us show now that the Bell inequalities cannot be proven without additional assumptions if we associate with particles in SPCE the functions acting on the orientation vectors of polarizers and if we use consequently the frequency definition of all measured probabilities and expectation values [12].

Let us assume that we have a beam of pairs of spin-1/2 particles or photons moving in opposite directions toward two ideal spin measuring devices Y and X characterized by the orientation vectors y and x, respectively. To make our argument as clear as possible, we deal here with 100% efficient devices, perfect correlation and angular resolution (a generalization to the real experimental conditions can be done without any problem). We associate with each particle a corresponding function s, s: $S^{(2)}\to\{-1, 1\}$, where $S^{(2)}$ is the set of unit orientation vectors, and with each pair of particles corresponding functions s and $s'=-s$ randomly chosen from a set F (a set of functions on $S^{(2)}$ satisfying some

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supplementary conditions to be specified later).

From experimental data we can estimate the expectation values of $E(y, x)$ closely related with the spin polarization correlation function. The estimated values of $E(y, x)$ are obtained by averaging the results of the different runs from the corresponding experiments. Let us assume, for simplicity, that in each experiment we analyze M runs of the same length N. The expectation value $E(y, x)=\lim_{N,M\to\infty} E_{MN}(y, x)$, where

$$E_{MN}(y, x)=\frac{1}{M}\sum_{j=1}^{M} r_{jN}(y, x) ,$$

$$r_{jN}(y, x)=-\frac{1}{N}\sum_{l=1}^{N} s_{lj}(y)s_{lj}(x) , \quad (1)$$

where $s_{lj}\in$ F is a function associated with the first particle of the lth pair of particles from the jth run. The experiment is characterized by a couple of vectors (y, x).

Let us observed that using Bell's method [6,8] we may prove the following relations for any value of $j=1, ..., M$,

$$-A_j=\sum_{l=1}^{N} s_{lj}(y)s_{lj}(x)-\sum_{l=1}^{N} s_{lj}(y)s_{lj}(x') , \quad (2)$$

$$-A_j=\sum_{l=1}^{N} s_{lj}(y)s_{lj}(x)[1-s_{lj}(x)s_{lj}(x')] , \quad (3)$$

$$|A_j/N|\leqslant 1-\frac{1}{N}\sum_{l=1}^{N} s_{lj}(x)s_{lj}(x') . \quad (4)$$

If we analyze the formulae (2)–(4) we see that *only* the first term in $-A_j/N$ is a quantity obtained from the data of the experiment (y, x). Whether one may relate the other terms in (4) with the data of real experiments remains an *open question*. In fact, the only thing we know is that s_{lj} are randomly chosen from the set F, thus in general $s_{lj}\neq s_{mk}$ if $l\neq m$ or $j\neq k$, where j and k denote different runs in the experiment (y, x) and l and m different elements of the jth and kth run, respectively. The same argument is valid, if we compare the results from different experiments.

Let us examine now what additional assumptions are needed to enable the proof of the Bell inequality from the formulae (1)–(4).

(1) Let us suppose that F contains only K different spin functions and that in each run we are sampling, with replacement, the functions s_{lj} from F. There are only K^N different sets containing N pairs of functions, thus a probability of finding a particular set in a given run is equal to K^{-N}. It is reasonable to assume that, if we make a sum of all the runs, the frequencies of the different sets will be similar in all the experiments. If we add all the inequalities (4) dividing by M and letting M tend to infinity we obtain the first Bell inequality

$$|E(y, x)-E(y, x')|\leqslant 1+E(x, x') . \quad (5)$$

(2) If F is infinite, we could have obtained the formula (5) from (1)–(4) if the experiments had been performed *in a way ensuring* the validity of the procedure described in point (1). For example if in each experiment we have the same final collection of the sets of N pairs of functions we obtain the inequality (5) directly for the empirical averages E_{MN},

$$|E_{MN}(y, x)-E_{MN}(y, x')|\leqslant 1+E_{MN}(x, x') . \quad (6)$$

It might seem plausible to accept that point (2) is valid for any F and for the experiments discussed above, but this is not so. Let us make the following remarks:

(a) If the set F contains uncountably many elements (F has the power of the continuum) the probability of drawing from F at random a particular spin function, if one is sampling with or without replacement *any* number of times, is equal to *zero*.

(b) A random experiment (y, x) does not consist in comparing different spin funtions, but it is designed to estimate four conditional probabilities $p(y^+, x^+)$, $p(y^-, x^+)$, $p(y^+, x^-)$ and $p(y^-, x^-)$, where $p(y^+, x^+)$ is the probability for the second particle of each pair of particles to be transmitted by the device X if the first is transmitted by the device Y. According to the model discussed above, $p(y^+, x^+)$ is equal to the probability that $s(x)=+1/2$, if spin functions s are drawn from $F_-(y)$ ($F_-(y)=\{s\in F, s(y)=-1/2\}$). A sample space S_{yx} for the experiment (y, x) contains only four points and the probabilities $p(y^\pm, x^\pm)$ can be estimated from the observed relative frequencies. A priori S_{yx} and $S_{y'x'}$ are unrelated for $y\neq y'$ or $x\neq x'$ because the random experiments (y, x) and (y', x') are different (similarly to the Bertrand situation).

(c) In general, if we deal with different independent random experiments, we cannot and *we do not* use a unique sample space to describe them all. Otherwise we would violate Kolmogorov's axioms of the probability calculus. On the contrary we use one sample space S to describe L different random experiments in which probability distributions of L random variables R_i, $i=1, ..., L$, are studied, if these experiments can be replaced by one random experiment designed to study the probability distribution of one $\tilde{L}$-dimensional random variable $\boldsymbol{R}=(\tilde{R}_1, \tilde{R}_2, ..., \tilde{R}_{\tilde{L}})$. From the data of this experiment information concerning any particular variable R_i can be obtained. A model of this type was used to prove the Bell inequalities.In fact, if one considers a set of m macroscopically distinguishable orientation vectors n_i of the spin measuring devices Y_i, one can perform m^2 experiments ($y=n_i$, $x=n_j$). Now if with each event one associates the supplementary parameters $\lambda=(\lambda_1, \lambda_2)$ and with each Y_i a bivalued function A_i ($A_i(\lambda_j)=\pm 1$), one can represent λ by a set of $2m$ integers $\lambda=(A_1(\lambda_1), A_1(\lambda_2); A_2(\lambda_1), A_2(\lambda_2);...; A_m(\lambda_1), A_m(\lambda_2))$. The overall sample space S is of dimension 2^{2m} and one may say that the state of each particle is completely described by the *simultaneously measurable* values of the spin projections n_i, $i=1, ..., m$. The various SPCE cannot be replaced by one random experiment of the type discussed above and in our opinion this is the reason why the Bell inequalities do not hold. The various probabilities appearing in their proofs are counterfactual and have nothing to do with the measured ones.

(d) If in the formula (1) one replaces s_{lj} by $2\tilde{s}_{lj}$ and by F_0 (where s_{lj} are Pitovsky spin funtions [3] and F_0 their set in the P-model), one can reproduce all QMP [3,12] without introducing faster-than-light influences.

To conclude, the Bertrand statement, that talking about probabilities we should always indicate the random experiment needed to estimate their values, is very important in epistemological disussions on the foundations of quantum theory.

Note added

After this paper had been completed our attention was drawn to two papers by de Baere [13] in which the author strongly claimed that the violation of the Bell inequalities is due to the non-reproducibility of a set of hidden variables in subsequent experiments. He was, however, unable to prove that if we repeat the experiment to measure the *same* spin correlations, we obtain consistent results nor that the agreement with the quantum mechanical predictions can be achieved. The use of the *uncountable* set of spin functions associated with particles (suggested for the first time by Pitovsky) together with the operational definition of all the probabilities gave the solution to these problems.

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