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RELATIVISTIC QUARK-MODEL PREDICTIONS FOR THE TRANSVERSITY AMPLITUDES

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Abstract: The relations between the transversity amplitudes following from the relativistic quark model for high-energy and small-angle scattering are obtained. The usual a-type relations are obtained but with fixed-t-dependent phenomenological parameters. The b- and c-type relations cannot be obtained without additional assumptions on the particle state vectors.

In recent papers [1, 2] the significance of relativistic effects in the consideration of scattering in the quark model was pointed out.

There are a large number of results of the non-relativistic quark model, in particular the polarization predictions [3, 4], so it is interesting to see how these predictions change, if we use the relativistic model from ref. [2].

Our considerations are analogous to those in ref. [2], but we proceed in a different manner and obtain compact formulae for the reaction amplitudes in a general case. These formulae enable the immediate derivation of the polarization predictions.

We shall consider the reactions of the type:

$$B + H_1 \rightarrow B^* + H_2 \,, \tag{1}$$

where B and B*denote baryons from $\frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decouplet and H_1 and H_2 are appropriate hadrons.

We represent mesons and baryons by the superpositions of the free quark states [2]:

$$|M\,0\,\mu\rangle = \int d\rho_M \sum_{\alpha_i, a_i} C_\mu^{a_1 a_2}(M; \alpha_1, \alpha_2)\,|\boldsymbol{q}_1 \alpha_1 a_1\rangle |\boldsymbol{q}_2 \alpha_2 a_2\rangle \,, \tag{2}$$

$$|B\,0\,\mu\rangle = \int d\rho_B \sum_{\alpha_i, a_i} C_\mu^{a_1 a_2 a_3}(B; \alpha_1, \alpha_2, \alpha_3)\,|\boldsymbol{q}_1, \alpha_1 a_1\rangle |\boldsymbol{q}_2 \alpha_2 a_2\rangle |\boldsymbol{q}_3 \alpha_3 a_3\rangle \,. \tag{3}$$

Here μ enumerates the spin states of the particle: α_i and a_i are spin and unitary spin variables of the ith quark; $C_\mu^{a_1 a_2}(M; \alpha_1, \alpha_2)$ and

$C_{\mu}^{a_1 a_2 a_3}(\mathrm{B}; \alpha_1, \alpha_2, \alpha_3)$ are the appropriate SU(6) Clebsch-Gordan coefficients. The free quark states are defined as follows:

$$|\boldsymbol{q}\,\alpha a\rangle = R(\hat{\boldsymbol{q}})L_x(q)R^{-1}(\hat{\boldsymbol{q}})\,|w\,0\,0\,0\,\alpha a\rangle\,, \tag{3}$$

where $|w\,0\,0\,0\,\alpha a\rangle$ is the rest frame state vector of the 'free quark' and w is its effective mass. The $R(\hat{\boldsymbol{q}})$ is the rotation operator which transforms the x-axis in the direction $\hat{\boldsymbol{q}}$ of the quark's momentum. The weight functions $\mathrm{d}\rho_{\mathrm{M}}$ and $\mathrm{d}\rho_{\mathrm{B}}$ are given by:

$$\mathrm{d}\rho_{\mathrm{M}} = \delta_3(\boldsymbol{q}_1+\boldsymbol{q}_2)f_{\mathrm{M}}(\boldsymbol{q}_1,\boldsymbol{q}_2)\mathrm{d}_3\boldsymbol{q}_1\,\mathrm{d}_3\boldsymbol{q}_2\,, \tag{4}$$

$$\mathrm{d}\rho_{\mathrm{B}} = \delta_3(\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{q}_3)f_{\mathrm{B}}(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3)\mathrm{d}_3\boldsymbol{q}_1\,\mathrm{d}_3\boldsymbol{q}_2\,\mathrm{d}_3\boldsymbol{q}_3\,, \tag{5}$$

where f_{M} and f_{B} are complex functions invariant under interchange of the vectors and arbitrary simultaneous rotations of all the vectors.

The expressions (2) and (3) represent the rest-frame state vectors of mesons and baryons as a superposition of 'free-quark' states [2]. We assume that quark momenta in the rest frame of the hadron are non-relativistic so the functions $f_{\mathrm{B}}(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3)$ and $f_{\mathrm{M}}(\boldsymbol{q}_1,\boldsymbol{q}_2)$ do not vanish only for q_i such that $|\boldsymbol{q}_i/w| \ll 1$. Since we neglect the terms of the order $|\boldsymbol{q}_i/w|^2$ the particle's mass is equal to the sum of the masses of the constituent quarks. To make the future calculations consistent we neglect the mass difference between different quarks, which implies that we also neglect the mass difference between different baryons and mesons. We use 'average' masses of baryons and mesons instead of their real masses. The assumptions given above imply that the quark mass is $w = \frac{1}{3}M_{\mathrm{B}} = \frac{1}{2}M_{\mathrm{M}}$.

We are going to find the c.m. transversity amplitudes of the reaction (1). For this reason we define the following states:

$$|\theta p\, m\rangle = R_z(\theta)L_x(p)R_z^{-1}(\theta)\,|0\,0\,m\rangle\,. \tag{6}$$

Here m enumerates the spin states of the particle, p is the modulus of its momentum, the z-axis is perpendicular to the reaction plane, and the x-axis is in the direction of the momentum of the first particle. The states (6) are related to the transversity states $|\theta p\,\mu\rangle$ in the following way:

$$|\theta p\,\mu\rangle = \sum_m \delta_{\mu m}\,\mathrm{e}^{-i\theta m}\,|\theta p\, m\rangle\,. \tag{7}$$

Let us present our considerations on the example of the reaction of the type

$$\mathrm{B} + \mathrm{M}_1 \to \mathrm{B}^* + \mathrm{M}_2\,, \tag{8}$$

where M_1 and M_2 are the appropriate mesons.

Now taking into account eqs. (2) and (3) we obtain from the formula (5) the following baryon-meson states:

$$|\mathrm{BM}\,\theta p m_i m_j\rangle = |\mathrm{B}\,\theta p m_i\rangle \otimes |\mathrm{M}\,\pi + \theta p m_j\rangle$$

$$= \int \mathrm{d}\rho_\mathrm{B}\, \mathrm{d}\rho_\mathrm{M} \sum_{\alpha_i, a_i} C_{m_i}^{a_1 a_2 a_3}(\mathrm{B}; \alpha_1, \alpha_2, \alpha_3) C_{m_j}^{a_4 a_5}(\mathrm{M}; \alpha_4, \alpha_5)$$

$$\times \bigotimes_{i=1}^{5} \sum_{\beta_i} |\boldsymbol{q}_i' \beta_i a_i\rangle D^{\frac{1}{2}}_{\beta_i \alpha_i}(\boldsymbol{q}_i) \ , \qquad (9)$$

where $D^{\frac{1}{2}}(\boldsymbol{q}_k)$ are the appropriate Wigner rotations. The explicit form of $D^{\frac{1}{2}}(\boldsymbol{q}_k)$ is the following:

$$D^{\frac{1}{2}}(\boldsymbol{q}_k) = \frac{2W_\mathrm{H}}{\sqrt{2W_\mathrm{H}(E_{q_k} + W_\mathrm{H}) 2M_\mathrm{H}(E_\mathrm{H} + M_\mathrm{H}) 2W_\mathrm{H}(E_{q_k'} + W_\mathrm{H})}}$$

$$\times \begin{cases} E_{q_k} M_\mathrm{H} + E_{q_k'} M_\mathrm{H} + W_\mathrm{H}(E_\mathrm{H} + M_\mathrm{H}) - i(\boldsymbol{q}_k \times \boldsymbol{P})\boldsymbol{\sigma} & \text{for} \quad k = 1, 2, 3 \ , \\ E_{q_k} M_\mathrm{H} + E_{q_k'} M_\mathrm{H} + W_\mathrm{H}(E_\mathrm{H} + M_\mathrm{H}) + i(\boldsymbol{q}_k \times \boldsymbol{P})\boldsymbol{\sigma} & \text{for} \quad k = 4, 5 \ . \end{cases} \qquad (10)$$

The notation is the following: $\boldsymbol{P}$ is the momentum of the baryon, $\boldsymbol{q}_k$ is the momentum of the kth quark in the rest frame of the appropriate hadron, M_H is the mass of the hadron, E_{q_k} is the energy of the kth quark in the rest frame of the corresponding hadron, $E_{q_k'}$ is the energy of the kth quark in the c.m. system, E_H is the energy of the hadron in the c.m. system, and W_H is the mass of the quark from the hadron H. The relation between the energy and momentum for quarks and particles is that imposed by special relativity. The vector $\boldsymbol{P}$ is chosen in the form

$$\boldsymbol{P} = p(\cos\theta, \sin\theta, 0) = p\hat{\boldsymbol{P}} \ . \qquad (11)$$

Neglecting the terms $|\boldsymbol{q}_k/W_\mathrm{H}|^2$, $|M/p|^2$, $|q_k M_\mathrm{H}/W_\mathrm{H} p|$ and the terms of higher orders in $\boldsymbol{q}_k$ we obtain:

$$D^{\frac{1}{2}}(\boldsymbol{q}_k) = \begin{cases} I - \dfrac{i}{2w_\mathrm{B}} (\boldsymbol{q}_k \times \hat{\boldsymbol{P}})\boldsymbol{\sigma} & \text{for} \quad k = 1, 2, 3 \ , \\ I + \dfrac{i}{2w_\mathrm{M}} (\boldsymbol{q}_k \times \hat{\boldsymbol{P}})\boldsymbol{\sigma} & \text{for} \quad k = 4, 5 \ . \end{cases} \qquad (12)$$

Now in a similar way (details in appendix A) as in ref. [2] we obtain the following expression for the amplitude $f_{m_3 m_4 m_1 m_2}$ of reaction (8):

$$f_{m_3 m_4 m_1 m_2} = \langle \mathrm{B}^* \mathrm{M}_2\, \theta\, m_3\, m_4 | T | \mathrm{B}\, \mathrm{M}_1\, 0\, m_1\, m_2 \rangle$$

$$\times \exp\left[-\tfrac{1}{2} ip \sin\theta \left(\frac{m_1+m_3}{M_\mathrm{B}} + \frac{m_2+m_4}{M_\mathrm{M}}\right)\right] f_{m_3 m_4 m_1 m_2}$$

$$= \sum_{\alpha_i, \beta_i} \sum_{a_i, b_i} C^{a_1 a_2 a_3}_{m_1} (\mathrm{B}; \alpha_1, \alpha_2, \alpha_3)\, C^{a_4 a_5}_{m_2} (\mathrm{M}_1; \alpha_4, \alpha_5)$$

$$\times C^{b_1 b_2 b_3}_{m_3} (\mathrm{B}^*; \beta_1, \beta_2, \beta_3)\, C^{b_4 b_5}_{m_4} (\mathrm{M}_2; \beta_4, \beta_5)$$

$$\times \Big(\langle b_1 \beta_1\, b_4 \beta_4 | a_1 \alpha_1\, a_4 \alpha_4 \rangle\, \delta_{a_2 b_2}\, \delta_{a_3 b_3}\, \delta_{a_5 b_5}\, \delta_{\alpha_2 \beta_2}\, \delta_{\alpha_3 \beta_3}\, \delta_{\alpha_5 \beta_5}$$

$$+ \langle b_2 \beta_2\, b_4 \beta_4 | a_2 \alpha_2\, a_4 \alpha_4 \rangle\, \delta_{a_1 b_1}\, \delta_{a_3 b_3}\, \delta_{a_5 b_5}\, \delta_{\alpha_1 \beta_1}\, \delta_{\alpha_3 \beta_3}\, \delta_{\alpha_5 \beta_5}$$

$$+ \langle b_3 \beta_3\, b_4 \beta_4 | a_3 \alpha_3\, a_4 \alpha_4 \rangle\, \delta_{a_1 b_1}\, \delta_{a_2 b_2}\, \delta_{a_5 b_5}\, \delta_{\alpha_1 \beta_1}\, \delta_{\alpha_2 \beta_2}\, \delta_{\alpha_5 \beta_5}$$

$$+ \langle b_1 \beta_1\, b_5 \beta_5 | a_1 \alpha_1\, a_5 \alpha_5 \rangle\, \delta_{a_2 b_2}\, \delta_{a_3 b_3}\, \delta_{a_4 b_4}\, \delta_{\alpha_2 \beta_2}\, \delta_{\alpha_3 \beta_3}\, \delta_{\alpha_4 \beta_4}$$

$$+ \langle b_2 \beta_2\, b_5 \beta_5 | a_2 \alpha_2\, a_5 \alpha_5 \rangle\, \delta_{a_1 b_1}\, \delta_{a_3 b_3}\, \delta_{a_4 b_4}\, \delta_{\alpha_1 \beta_1}\, \delta_{\alpha_3 \beta_3}\, \delta_{\alpha_4 \beta_4}$$

$$+ \langle b_3 \beta_3\, b_5 \beta_5 | a_3 \alpha_3\, a_5 \alpha_5 \rangle\, \delta_{a_1 b_1}\, \delta_{a_2 b_2}\, \delta_{a_4 b_4}\, \delta_{\alpha_1 \beta_1}\, \delta_{\alpha_2 \beta_2}\, \delta_{\alpha_4 \beta_4} \Big) . \qquad (13)$$

Here M_B denotes the masses of the baryons B and B*, and M_M denotes the masses of the mesons M_1 and M_2. The $\langle b_i \beta_i\, b_j \beta_j | a_k \alpha_k\, a_l \alpha_l \rangle$ is an averaged quark-quark amplitude:

$$\langle b_i \beta_i\, b_j \beta_j | a_k \alpha_k\, a_l \alpha_l \rangle = \exp\left[-\tfrac{1}{2} ip \sin\theta \left(\frac{\alpha_k+\beta_i}{M_\mathrm{B}} + \frac{\alpha_l+\beta_j}{M_\mathrm{M}}\right)\right]$$

$$\times \int \mathrm{d}\rho_{\mathrm{BM}} \sum_{\gamma_i} D^{\frac{1}{2}}_{\gamma_3 \beta_i}(q_3)^{\mathrm{f}}\, D^{\frac{1}{2}}_{\gamma_4 \beta_j}(q_4)^{\mathrm{f}}\, D^{\frac{1}{2}}_{\gamma_1 \alpha_k}(q_1)^{\mathrm{i}}\, D^{\frac{1}{2}}_{\gamma_2 \alpha_l}(q_2)^{\mathrm{i}}$$

$$\times \langle q_3\, q_4\, \gamma_3\, \gamma_4\, b_i\, b_j | \tau | q_1\, q_2\, \gamma_1\, \gamma_2\, a_k\, a_l \rangle , \qquad (14)$$

where $D^{\frac{1}{2}}(q_k)^{\mathrm{i,f}}$ are defined

$$D^{\frac{1}{2}}(q_k)^{\mathrm{i,f}} = \begin{cases} I - \dfrac{i}{2w_{\mathrm{B}}}(q_k \times \hat{P}^{\mathrm{i,f}})\sigma & \text{for} \quad k = 1,3 \,, \\ I + \dfrac{i}{2w_{\mathrm{M}}}(q_k \times \hat{P}^{\mathrm{i,f}})\sigma & \text{for} \quad k = 2,4 \,, \end{cases} \tag{15}$$

where

$$\hat{P}^{\mathrm{i}} = (1,0,0) \,, \tag{16}$$

$$\hat{P}^{\mathrm{f}} = (\cos\theta,\ \sin\theta, 0) \,. \tag{17}$$

Now taking into account eqs. (7) and (13) and the results of ref. [4] we obtain the following relations between the transversity amplitudes of the reaction (8):

$$f_{\mu_3\mu_4\mu_1\mu_2} = \exp\left[-i\left(\frac{p}{M_{\mathrm{B}}}\sin\theta + \theta\right)\right] N_{\frac{3}{2}}(\mu_3) N_{\frac{1}{2}}(\mu_1) f_{\mu_3+1\,\mu_4\,\mu_1+1\,\mu_2}$$
$$+ \exp\left[i\left(\frac{p}{M_{\mathrm{B}}}\sin\theta + \theta\right)\right] N_{\frac{3}{2}}(-\mu_3) N_{\frac{1}{2}}(-\mu_1) f_{\mu_3-1\,\mu_4\,\mu_1-1\,\mu_2} \,. \tag{18}$$

The coefficients $N_{\frac{3}{2}}(\mu)$ and $N_{\frac{1}{2}}(\mu)$ are defined [4]:

$$N_{\frac{3}{2}}(\mu) = \begin{cases} \dfrac{1}{\sqrt{3}} & \text{for} \quad \mu = \tfrac{1}{2} \,, \\ 1 & \text{for} \quad \mu = -\tfrac{1}{2} \,, \\ \sqrt{3} & \text{for} \quad \mu = -\tfrac{3}{2} \,, \\ 0 \text{ in all other cases} \,, \end{cases} \tag{19}$$

$$N_{\frac{1}{2}}(\mu) = \begin{cases} 1 & \text{for} \quad \mu = -\tfrac{1}{2} \,, \\ 0 \text{ in all other cases} \,. \end{cases} \tag{20}$$

The relations (18) are also valid for the reactions of the type (1).

The averaged quark-quark amplitudes (14) satisfy the same relations following from the parity conservation as those satisfied by the transversity amplitudes *. Under time reversal they behave in a different way. Using the time reversal and rotational invariance of the qq amplitudes one cannot obtain without additional assumptions the b- and c-type relations for the transversity amplitudes of the reaction (1) for $\theta \neq 0$. The b- and c-type relations can be obtained for $\theta = 0$, but in this case the spin-flip vanish, so these relations are trivially satisfied. One can imagine that for very

* The proofs of this and the following statements are given in appendix B.

small θ, such that $p \sin\theta \ll |q_i|$ they can be also valid. Then the b- and c-type relations are of the form:

$$f_{\mu_3\mu_4\mu_1\mu_2} = (-1)^N \exp\left[ip\sin\theta\left(\frac{\mu_1+\mu_3}{M_B} + \frac{\mu_2+\mu_4}{M_M}\right) + i\theta(\mu_1+\mu_2+\mu_3+\mu_4)\right] \times f_{-\mu_3-\mu_4-\mu_1-\mu_2}, \qquad (20)$$

for $\mu_1 - \mu_3 = \mu_4 - \mu_2 \neq 0$ in the case of the b-type relations and for $|\mu_1 - \mu_3| = |\mu_2 - \mu_4| \neq 0$ in the case of the c-type relations. Here N denotes the total number of baryons and pseudoscalar mesons participating in the reaction (1). The non-trivial b- and c-type relations can be exactly derived for the reaction $M_1 + V \to M_2 + V$, where V denotes a vector meson. Besides the fact that such reactions are not observed, the use of the vector-dominance model can give relations for the processes $M_1 + \gamma \to M_2 + \gamma$.

At the end we shall summarize and discuss our results.

We have used in a consequent way some particular additivity assumption. This assumption does not depend on the Lorentz frame in which the calculations are performed so we do not have the freedom of the choice of the phenomenological parameters, namely the rotation angles defining the 'additivity frame'.

It was shown that only for small-t one obtains the a-type relations for the transversity amplitudes which are identical to the usual relations but with fixed and explicitly given phenomenological parameters. Their t-dependence is not linear so testing of these predictions cannot be made by the simple averaging over all t and s. Moreover for the larger t one cannot without additional assumptions obtain the a-type relations. The use of the exact Wigner rotations (10) would lead to the replacement of the spectator Kronecker δ by some q_k dependent matrices, and so the breakdown of the usual predictions.

We have shown that the b- and c-type relations cannot be derived exactly for $\theta \neq 0$ without additional assumptions. However, for very small such that $-t \ll |q_k|$ they can be expected to be approximately satisfied but also with fixed phenomenological parameters. The region of the validity of these relations can be probably enlarged by a suitable choice of the weights (4) and (5) or additional assumptions on the qq amplitudes.

Although the simplified model considered here can have nothing to do with reality we should like to stress once more [4] that one should test the b- and c-type relations more precisely.

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APPENDIX A

We shall sketch here the derivation of the formula (13). Using the kinematical relations for the momenta of the spectator quarks [2] in our approximation

$$-t_1 = q_5 - q_7 = \frac{W_M}{M_B} p(p^i - p^f) = q_6 - q_8 ,$$

$$-t_2 = q_9 - q_{10} = -\frac{W_M}{M_M} p(p^i - p^f) \tag{A.1}$$

and the additivity assumption we obtain the following equality:

$$f_{m_3m_4m_1m_2} = \sum_{\alpha_i,\beta_i} \sum_{a_i,b_i} C^{a_1a_2a_3}_{m_1}(\mathrm{B};\alpha_1,\alpha_2,\alpha_3) C^{a_4a_5}_{m_2}(\mathrm{M}_1;\alpha_4\,\alpha_5)$$

$$\times C^{b_1b_2b_3}_{m_3}(\mathrm{B}^*;\beta_1,\beta_2,\beta_3) C^{b_4b_5}_{m_4}(\mathrm{M}_2;\beta_4,\beta_5) \int \mathrm{d}\rho_\mathrm{B}\,\mathrm{d}\rho_{\mathrm{M}_1}\,\mathrm{d}\rho_{\mathrm{B}^*}\,\mathrm{d}\rho_{\mathrm{M}_2}$$

$$\times D^\dagger_{\beta_1\beta_1'}(q_5) D^\dagger_{\beta_2\beta_2'}(q_6) D_{\alpha_1'\alpha_1}(q_7) D_{\alpha_2'\alpha_2}(q_8) \delta_{\alpha_1'\beta_1'}\,\delta_{\alpha_2'\beta_2'}\,\delta_{a_1b_1}\,\delta_{a_2b_2}$$

$$\times \delta_3(q_5 - q_7 + t_1)\delta_3(q_6 + q_8 + t_1) D^\dagger_{\beta_4\beta_4'}(q_9) D_{\alpha_4'\alpha_4}(q_{10}) \delta_{\alpha_4'\beta_4'}\,\delta_{a_4b_4}$$

$$\times \delta_3(q_9 - q_{10} + t_2) D^\dagger_{\beta_3\gamma_3}(q_3) D^\dagger_{\beta_5\gamma_4}(q_4)$$

$$\times \langle q_3\, q_4\, \gamma_3\, \gamma_4\, b_3\, b_4 | \tau | q_1\, q_2\, \gamma_1\, \gamma_2\, b_1\, b_2 \rangle D_{\gamma_1\alpha_3}(q_1) D_{\gamma_2\alpha_5}(q_2)$$

$$+ \text{other terms} . \tag{A.2}$$

Now using the relation for the approximate Wigner rotations (11):

$$D^\dagger_{\beta_ik}(q_5) D_{k\alpha_i}(q_7) = D_{\beta_i\alpha_i}\left(-\hat{n}\frac{p \sin\theta}{M_\mathrm{B}}\right) = \delta_{\beta_i\alpha_i} \exp\left[i\,\frac{p\sin\theta}{2M_\mathrm{B}}\,(\alpha_i + \beta_i)\right] , \tag{A.3}$$

we obtain

$$f_{m_3m_4m_1m_2} = \sum_{\alpha_i,\beta_i} \sum_{a_i,b_i} C^{a_1a_2a_3}_{m_1}(\mathrm{B},\alpha_1\,\alpha_2\,\alpha_3) \cdots C^{b_4b_5}_{m_4}(\mathrm{M}_2,\beta_4\,\beta_5)$$

$$\times \exp\left[\frac{ip\sin\theta}{2M_\mathrm{B}}\,(\alpha_1 + \beta_1 + \alpha_2 + \beta_2) + \frac{ip\sin\theta}{2M_\mathrm{M}}\,(\alpha_2 + \beta_2)\right] \delta_{\beta_1\alpha_1}\,\delta_{\beta_2\alpha_2}$$

$$\times \delta_{b_1a_1}\,\delta_{b_2a_2} \int \mathrm{d}\rho_\mathrm{BM}\, D^\dagger_{\beta_3\gamma_3}(q_3) \cdots D_{\gamma_2\alpha_5}(q_2) + \text{other terms} . \tag{A.4}$$

From eq. (A.4) one immediately obtains eqs. (13) and (14) with $\mathrm{d}\rho_\mathrm{BM}$ written explicitly in formula (B.11) of appendix B.

Had we used the exact Wigner rotations (10) instead of the approximate ones (11), the product of the matrices $D^{\dagger}(\boldsymbol{q}_5)D(\boldsymbol{q}_1)$ would have the following form

$$D_{\beta_i k}(\boldsymbol{q}_5)D_{k\alpha_i}(\boldsymbol{q}_7) = A_{\beta_i\alpha_i}(\boldsymbol{q}_5, \boldsymbol{q}_7) , \qquad \text{(A.5)}$$

where $A_{\beta_i\alpha_i}(\boldsymbol{q}_5, \boldsymbol{q}_7)$ is non-diagonal, $\boldsymbol{q}_i$-dependent matrix. In that case, without additional assumptions, the amplitudes $f_{\mu_3\mu_4\mu_1\mu_2}$ with $|\mu_3 - \mu_1| > 1$ would not vanish, which is in contradiction with a-type relations.

APPENDIX B

We shall investigate the properties of the averaged qq amplitudes (13). We shall write these amplitudes in a different but equivalent way:

$$\langle b_1\beta_1\, b_4\beta_4 | a_1\alpha_1\, a_4\alpha_4\rangle = C \int d\rho_{\mathrm{BM}} \langle \boldsymbol{q}_3\boldsymbol{q}_4\, b_1\beta_1\, b_4\beta_4 | \tau | \boldsymbol{q}_1\boldsymbol{q}_2\, a_1\alpha_1\, a_4\alpha_4\rangle , \qquad \text{(B.1)}$$

where C is the same exponential factor as in eq. (13) and

$$\langle \boldsymbol{q}_3\boldsymbol{q}_4\, b_1\beta_1\, b_4\beta_4 | \tau | \boldsymbol{q}_1\boldsymbol{q}_2\, a_1\alpha_1\, a_4\alpha_4\rangle$$

$$= \langle p, \theta, \boldsymbol{q}_3, b_1, \beta_1; p, \pi+\theta, \boldsymbol{q}_4, b_4\beta_4 | \tau | p, 0, \boldsymbol{q}_1, a_1, \alpha_1; p, \pi, \boldsymbol{q}_2, a_2, \alpha_2\rangle , \qquad \text{(B.2)}$$

where

$$|p, \theta, \boldsymbol{q}, d, \gamma\rangle = R_z(\theta)L_x(p)R_z^{-1}(\theta)R(\hat{\boldsymbol{q}})L_x(q)R^{-1}(\hat{\boldsymbol{q}})|W, 0, 0, 0, d, \gamma\rangle . \qquad \text{(B.3)}$$

The operator $R(\boldsymbol{q})$ is the operator representing the rotation turning the x-axis into the direction of the vector $\boldsymbol{q}$. To derive relations between $\langle b_1\beta_1\, b_4\beta_4 | a_1\alpha_1\, a_4\alpha_4\rangle$ stemming from the parity conservation and the time reversal invariance of τ we shall notice that

$$P|p, \theta, \boldsymbol{q}, d, \gamma\rangle = \eta R_z(\theta+\pi)L_x(p)R_z^{-1}(\pi+\theta)$$

$$\times R(\hat{\boldsymbol{q}})R_z(\pi)L\ \ (q)R_z^{-1}(\pi)R^{-1}(\hat{\boldsymbol{q}})|W, 0, 0, 0, d, \gamma\rangle . \qquad \text{(B.4)}$$

Using eq. (B.4) and the rotational invariance of τ we obtain:

$$\langle \boldsymbol{q}_3\boldsymbol{q}_4\, b_1\beta_1\, b_2\beta_2 | \tau | \boldsymbol{q}_1\boldsymbol{q}_2\, a_1\alpha_1\, a_2\alpha_2\rangle = \eta_1\eta_2\eta_3\eta_4$$

$$\times (-1)^{\alpha_1+\alpha_2-\beta_1-\beta_2} \langle \boldsymbol{q}_3'\boldsymbol{q}_4'\, b_1\beta_1\, b_2\beta_2 | \tau | \boldsymbol{q}_1'\boldsymbol{q}_2'\, a_1\alpha_1\, a_2\alpha_2\rangle , \qquad \text{(B.5)}$$

where

$$\boldsymbol{q}_i = -R_z(-\pi)\boldsymbol{q}_i . \qquad \text{(B.6)}$$

Now let us notice that:

$$T\,|p,\theta,\boldsymbol{q},d,\gamma\rangle = R_z(\theta)R_y(\pi)L_x(p)R_y^{-1}(\pi)R_z^{-1}(\theta)$$

$$\times R(\hat{\boldsymbol{q}})R_y(\pi)L_x(q)R_y^{-1}(\pi)R^{-1}(\hat{\boldsymbol{q}})(-1)^{s-\gamma}\,|W,0,0,0,d,-\gamma\rangle\ . \qquad \text{(B.7)}$$

Taking into account the equalities:

$$T^+\tau T = \tau^+\ ,$$

$$\langle \boldsymbol{q}_3\boldsymbol{q}_4 b_1\beta_1 b_2\beta_2|T^+\tau T|\boldsymbol{q}_1\boldsymbol{q}_2 a_1\alpha_1 a_2\alpha_2\rangle$$

$$= \overline{\langle \boldsymbol{q}_3\boldsymbol{q}_4 b_1\beta_1 b_2\beta_2|\tau|\boldsymbol{q}_1\boldsymbol{q}_2 a_1\alpha_1 a_2\alpha_2\rangle} \qquad \text{(B.8)}$$

and using the rotational invariance of τ we obtain from eq. (B.7) the following relations:

$$\langle \boldsymbol{q}_3\boldsymbol{q}_4 b_1\beta_1 b_2\beta_2|\tau|\boldsymbol{q}_1\boldsymbol{q}_2 a_1\alpha_1 a_2\alpha_2\rangle$$

$$= \exp[-i\theta(\beta_1+\beta_2-\alpha_1-\alpha_2)]\,\langle\tilde{\boldsymbol{q}}_1\tilde{\boldsymbol{q}}_2 a_1\alpha_1 a_2\alpha_2|\tau|\tilde{\boldsymbol{q}}_3\tilde{\boldsymbol{q}}_4 b_1\beta_1 b_2\beta_2\rangle\ , \qquad \text{(B.9)}$$

where

$$\tilde{\boldsymbol{q}}_i = -R_z(\theta)R_y(-\pi)\boldsymbol{q}_i\ . \qquad \text{(B.10)}$$

One way for the derivation of the relations for the averaged qq amplitudes implied by the relations (B.5) and (B.9) is to investigate the invariance of the measure $d\rho_{BM}$ under change of the variables $\boldsymbol{q}_i \to \boldsymbol{q}_i'$ and $\boldsymbol{q}_i \to \boldsymbol{q}_i''$. To do this we write more explicitly $d\rho_{BM}$. After some simple integrations with δ_3 functions coming from the scalar products of the spectator state vectors we obtain

$$d\rho_{BM} = [\int f_B(-\boldsymbol{q}_5-\boldsymbol{q}_1,\boldsymbol{q}_5,\boldsymbol{q}_1)f_{B^*}(-\boldsymbol{q}_5-\boldsymbol{q}_3-\boldsymbol{q}_1,\boldsymbol{q}_5+\boldsymbol{q}_1,\boldsymbol{q}_3)d_3\boldsymbol{q}_5]$$

$$\times f_{M_1}(-\boldsymbol{q}_2,\boldsymbol{q}_2)f_{M_2}(-\boldsymbol{q}_4,\boldsymbol{q}_4)\delta_3(\boldsymbol{q}_1-\boldsymbol{q}_3-2\boldsymbol{q}_1)$$

$$\times\,\delta_3(\boldsymbol{q}_2-\boldsymbol{q}_4-\boldsymbol{q}_2)d_3\boldsymbol{q}_1\,d_3\boldsymbol{q}_2\,d_3\boldsymbol{q}_3\,d_3\boldsymbol{q}_4\ , \qquad \text{(B.11)}$$

where

$$\boldsymbol{t}_1 = \frac{W}{M_B}(\boldsymbol{P}^f-\boldsymbol{P}^i)\ , \qquad \boldsymbol{t}_2 = \frac{W}{M_M}(\boldsymbol{P}^i-\boldsymbol{P}^f)\ .$$

Now using the properties of the f-functions one can see from eq. (B.11) that $d\rho_{BM}$ is invariant under change of variables (B.5) $\boldsymbol{q}_i \to \boldsymbol{q}_i$, $\boldsymbol{t}_i \to \boldsymbol{t}_i' = \boldsymbol{t}_i$. From this fact stems immediately that

$$\langle b_1\beta_1\, b_2\beta_2 \,|\, a_1\alpha_1\, a_2\alpha_2\rangle = \eta_1\eta_2\eta_3\eta_4(-1)^{\alpha_1+\alpha_2-\beta_1-\beta_2}\langle b_1\beta_1\, b_2\beta_2 \,|\, a_1\alpha_1\, a_2\alpha_2\rangle \,, \tag{B.12}$$

so the averaged qq amplitudes satisfy the same relations stemming from the parity conservation as the usual transversity amplitudes do. However, the $\mathrm{d}\rho_{\mathrm{BM}}$ is not invariant under the following change of variables:

$$\boldsymbol{q}_1 \to \boldsymbol{q}_1'' = \tilde{\boldsymbol{q}}_3\,, \qquad \boldsymbol{q}_2 \to \boldsymbol{q}_2'' = \tilde{\boldsymbol{q}}_4\,, \qquad \boldsymbol{q}_3 \to \boldsymbol{q}_3'' = \tilde{\boldsymbol{q}}_1\,,$$

$$\boldsymbol{q}_4 \to \boldsymbol{q}_4'' = \tilde{\boldsymbol{q}}_2\,, \qquad \boldsymbol{q}_5 \to \boldsymbol{q}_5'' = \tilde{\boldsymbol{q}}_5\,, \qquad \boldsymbol{t}_i \to \boldsymbol{t}_i'' = \tilde{\boldsymbol{t}}_i = -\boldsymbol{t}_i\,. \tag{B.13}$$

Using the properties of the f-functions we obtain from eqs. (B.11) and (B.13) the following relation

$$\langle b_1\beta_1\, b_2\beta_2 \,|\, a_1\alpha_1\, a_2\alpha_2\rangle = C\,\exp[-i\theta(\beta_1+\beta_2-\alpha_1-\alpha_2)]$$

$$\times \int \mathrm{d}\tilde{\rho}_{\mathrm{BM}}\,\langle \boldsymbol{q}_3''\,\boldsymbol{q}_4''\,a_1\alpha_1\,a_2\alpha_2 | \tau | \boldsymbol{q}_1''\,\boldsymbol{q}_2''\,b_1\beta_1\,b_2\beta_2\rangle\,, \tag{B.14}$$

where

$$\mathrm{d}\rho_{\mathrm{BM}} = \left[\int f_{\mathrm{B}}(-\boldsymbol{q}_5''-\boldsymbol{q}_3'', \boldsymbol{q}_5'', \boldsymbol{q}_3'')\, f_{\mathrm{B}*}(-\boldsymbol{q}_5''-\boldsymbol{q}_1''+\boldsymbol{q}_1, \boldsymbol{q}_5''-\boldsymbol{q}_1, \boldsymbol{q}_1'')\,\mathrm{d}_3\boldsymbol{q}_5''\right]$$

$$\times f_{\mathrm{M}_1}(-\boldsymbol{q}_4'', \boldsymbol{q}_4'')\, f_{\mathrm{M}_2}(-\boldsymbol{q}_2'', \boldsymbol{q}_2'')\,\delta_3(\boldsymbol{q}_1''-\boldsymbol{q}_3''-2\boldsymbol{t}_1)$$

$$\times\, \delta_3(\boldsymbol{q}_2''-\boldsymbol{q}_4''-\boldsymbol{t}_2)\,\mathrm{d}_3\boldsymbol{q}_1''\,\mathrm{d}_3\boldsymbol{q}_2''\,\mathrm{d}_3\boldsymbol{q}_3''\,\mathrm{d}_3\boldsymbol{q}_4''\,. \tag{B.15}$$

The $\mathrm{d}\tilde{\rho}_{\mathrm{BM}}$ is equal to $\mathrm{d}\rho_{\mathrm{BM}}$ only for $\boldsymbol{t}_1 = 0$, if $f_{\mathrm{B}} = f_{\mathrm{B}*}$ and $f_{\mathrm{M}_1} = f_{\mathrm{M}_2}$, since all f-functions are not translational invariant. If $\boldsymbol{t}_1$ is very small in comparison with the $\boldsymbol{q}_i$, one can expect that the replacement $\mathrm{d}\tilde{\rho}_{\mathrm{BM}}$ by $\mathrm{d}\rho_{\mathrm{BM}}$ will not make much difference, which implies the following approximate result:

$$\langle b_1\beta_1\, b_2\beta_2 \,|\, a_1\alpha_1\, a_2\alpha_2\rangle = \exp[-i\theta(\beta_1+\beta_2-\alpha_1-\alpha_2)]\,\langle a_1\alpha_1\, a_2\alpha_2 \,|\, b_1\beta_1\, b_2\beta_2\rangle\,. \tag{B.16}$$

The relations (20) are immediately obtained from eqs. (B.16) and (20) and from the results of ref. [4]. Analogous relations to (B.16) were usually assumed to obtain the b- and c-type relations [3,4].

One should also notice that $\mathrm{d}\tilde{\rho}_{\mathrm{MM}} = \mathrm{d}\rho_{\mathrm{MM}}$, if f_{M} does not depend on the kind of the meson.

From the relations (B.14) one also obtains the proper relations between the reaction amplitudes $f_{m_3m_4m_1m_2}$ and the reaction amplitudes $g_{m_1m_2m_3m_4}$ of the time-reversed reaction.

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