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ENTANGLEMENT AND BELL INEQUALITIES

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Abstract

The entangled quantum states play a key role in quantum information. The association of the quantum state vector with each individual physical system in an attributive way is a source of many paradoxes and inconsistencies. The paradoxes are avoided if the purely statistical interpretation of the quantum-state vector is adopted. According to the statistical interpretation, the quantum theory does not provide any deterministic prediction for any individual experimental result obtained for a free physical system, for a trapped ion, or for a quantum dot. In this article, it is shown that, if the statistical interpretation is used, then contrary to the general belief, the quantum theory does not predict for the ideal spin singlet state perfect anti-correlation of the coincidence counts for the distant detectors. Subsequently the various proofs of the Bell's theorem are reanalyzed and, in particular, the importance and implications of using the unique probability space in these proofs are elucidated. The use of the unique probability space is shown to be equivalent to the use of joint probability distributions for noncommuting observables. The experimental violation of the Bell's inequalities proves that the naive realistic particle, like the spatio-temporal description of various quantum mechanical experiments, is impossible. Of course, it does not give any argument for the action at a distance and it does not provide the proof of the completeness of quantum mechanics. The fact that the quantum-state vector is not an attribute of a single quantum system and that the quantum observables are contextual has to be taken properly into account in any implementation of the quantum computing device.

Keywords: Entanglement, Bell's inequalities, quantum information, quantum computing, Einstein–Podolsky–Rosen correlations, quantum cryptography.

1. Introduction

The long-range nonclassical correlations characterizing the entangled quantum states are at the base of the quantum computer project [1–3], state teleportation, and quantum cryptography [4–6]. The mathematical structure and possible time evolutions of the quantum states have been studied, and considerable progress has been achieved [7–9]. The entanglement witnesses have been constructed which may help to distinguish between different entangled states in the experiment [10, 11]. Quantum states and quantum process tomography have been studied and experimentally implemented [1, 12–15]. In spite of this incontestable progress of quantum information in some papers, the state vectors (qubits) are treated as attributes of the individual quantum system which can be manipulated and modified quasi-instantaneously. One may also occasionally find the picture of the Schrödinger cat and hear a story of the twin point-like particles communicating at a distance with faster than light signals. It seems that the abstract, statistical, and contextual character of the quantum description of Nature is sometimes

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forgotten. Besides it is usually assumed that a single measurement reduces instantaneously the state vector of a physical system.

Problems related to the quantum theory of measurement and the notion of the state-vector reduction have for decades been a subject of discussion between people interested in the foundations of quantum theory, and still there is no unanimity. The most consistent seems to us the point of view of followers of the so-called purely statistical interpretation of quantum theory, which evolved from the interpretation advocated for the first time by Einstein [16, 17]. According to the statistical interpretation, the pure state vector Ψ or the density matrix ϱ describes only the statistical properties of an ensemble of similarly prepared systems. For the trapped ions and quantum dots, it describes the statistical properties of repeated measurements on the same ion or the same quantum dot after the same initial preparation. The statistical interpretation was extensively discussed by Ballentine [18]. In his already classic textbook on quantum mechanics based on the statistical interpretation, we may read [19]: "Once acquired, the habit of considering an individual particle to have its own wave function is hard to break. Even though it has been demonstrated to be strictly incorrect, it is surprising how seldom it leads to a serious error." In the statistical interpretation, the state-vector reduction is a passage from the description of a whole ensemble to the description of a sub-ensemble obtained from the initial ensemble by so-called nondestructive measurements. The important additional arguments in favor of the statistical interpretation have been given recently by Allaverdyan, Balian, and Nieuwenhuizen [20].

Since most of the predictions of quantum theory are of statistical nature, the famous Einstein–Podolsky–Rosen question [16] might be asked, whether and in what sense the quantum theory provides a complete description of the individual physical system. In fact, the statistical interpretation leaves, in principle, a place for introducing the supplementary parameters (called often hidden variables) which would determine the behavior of each particular physical system during the experiment. Several theories with supplementary parameters have been discussed [21]. The most influential was the paper by Bell [22], who analyzed a large family of theories with supplementary parameters (the so-called local or realistic hidden variable theories) and showed that their predictions must violate, for some configurations of the experimental set-up, the quantum mechanical predictions for spin-polarization-correlation experiments dealing with pairs of electrons or photons produced in a singlet state. Bell's argument was put into experimentally verifiable form by Clauser, Horne, Shimony, and Holt [23]. Several experiments, in particular those by Aspect et al. [24, 25] (see, also [26]), confirmed the predictions of quantum mechanics. The general conclusion summarized in the excellent review by Clauser and Shimony [27] was that, if one wants to understand the experimental data, "either one must totally abandon the realistic philosophy of most working scientists or dramatically revise our concept of space time," which unwillingly encouraged speculations about a spooky action at a distance.

It was shown by many authors that the assumptions made in local or realistic hidden variable theories were more restrictive and questionable than they seemed to be and the Bell's inequalities may be violated not only by quantum experiments but also by macroscopic ones [28–33]. The recent experiments seemed to close the remaining loopholes [34,35], but the violation of the Clauser–Horne–Shimony–Holt inequality may be consider neither as proof of the completeness of quantum mechanics nor an indication of faster-than-light communication [7, 14, 36–39]. An extensive discussion of the concept of probability was given by Khrennikov [40] and Holevo [41]. The role of the contextuality and remaining loopholes in the Bell's proof were recently underlined by Khrennikov and Volovich [42–44].

In this short paper we refine and complement some of our old arguments and forgotten ideas [30–33], hoping that it can shed some light on the problems we face in quantum information.

The paper is organized as follows.

In Sec. 2, we reanalyze, in view of the statistical interpretation, the properties of entangled idealized spin singlet state. In particular, we show that there is no prediction for perfect correlations of counts for the faraway detectors and no Einstein–Podolsky–Rosen–Bohm paradox. Let us underline that this lack of perfect correlations is of a deeper nature than the lack of perfect correlations in all real experiments, which is attributed to the decoherence, experimental systematic and statistical errors, and efficiency of detectors [14, 27, 39]. In Sec. 3, we analyze some proofs of the Bell's and Clauser–Horne–Shimony–Holt inequalities, clearly demonstrating that the use of the unique probability space is equivalent to the use of joint probability distributions for noncommuting observables or to the assumption that all random variables corresponding to the physical observables studied are completely independent, thus uncorrelated. Let us note that most of the proofs of the recent generalizations of the Clauser–Horne–Shimony–Holt inequalities to the n qubits are usually done assuming the factorization of the expectation functions, thus the statistical independence of the corresponding random variables.

2. A Singlet State

Let us state the essential points of the Einstein–Podolsky–Rosen–Bohm reasoning, using the notation and phrasing from [19].

The singlet spin state vector for a system of two particles has the form

$$\Psi_0 = (\mid +\rangle \otimes \mid -\rangle - \mid -\rangle \otimes \mid +\rangle)\,\sqrt{1/2}, \tag{1}$$

where the single particle vectors $\mid +\rangle$ and $\mid -\rangle$ denote "spin up" and "spin down" with respect to some coordinate system.

(i) Even if the orbital state is not stationary, the interactions do not involve spin and so the spin states will not change.

(ii) The particles are allowed to separate and, when they are well beyond the range of the interaction, we can measure the z component of spin of particle 1.

(iii) Because the total spin is zero, we can predict with certainty, and without in any way disturbing the second particle, that the z component of spin of particle 2 must have the opposite value. Thus the value of $\sigma_z^{(2)}$ is an element of reality, according to the Einstein–Podolsky–Rosen criterion.

(iv) The singlet state is invariant under rotation and it has the same form (1) in term of "spin up" and "spin down" if the directions "up" and "down" are referred to any other axis. Thus, following the Einstein–Podolsky–Rosen criterion, we may argue that the values of $\sigma_x^{(2)}$, $\sigma_y^{(2)}$, and any number of other spin components are also elements of reality for particle 2.

What is wrong with this argument?

In (i) all possible decoherence due to the interaction with the environment is neglected.

In (ii), by saying that the particles had a time to separate, we assume a mental image of two point-like particles, which are produced and which after some time become separated and free.

Even if we assume that items (i) and (ii) are correct, then item (iii) is wrong and it will be proven below, in view of the statistical interpretation.

We do not see any particular couple of the particles and we do not follow its space–time evolution. We record only the clicks on the faraway coincidence counters. To be able to deduce the value of a particular spin projection for particle 2 from the measurement made on particle 1, we should have had for each experiment (A, B) a different experimental design (impossible to realize) giving us much more information on each couple of the particles than we have in a simple coincidence experiment. Similar arguments were given by Bohr [45, 46] in his neither well understood nor frequently read answer to the original paper by Einstein, Podolsky, and Rosen.

We interpret a click as a detection of the particle which passed by a polarization filter and which was registered by a detector. According to the statistical interpretation, only an ensemble of these particles is described by the one-particle state vector $|+\rangle$ or $|-\rangle$ with respect to the axis determined by the filter.

Let us note that if (iii) is not correct than (iv) does not follow and there is no Einstein–Podolsky–Rosen paradox. According to the statistical interpretation, a state Ψ_0 allows one only to find the statistical correlations observed in a long run of various experiments with different couples (A, B) of the spin polarization analyzers characterized by macroscopic direction vectors $\mathbf{A}$ and $\mathbf{B}$. Since the angle between $\mathbf{A}$ and $\mathbf{B}$ is a continuous variable, the quantum theory gives us the probability density functions not the probabilities. Let us go back to the mathematical formalism of the quantum theory.

Let $\sigma_{\mathbf{a}} = \sigma \cdot \mathbf{a}$ denote the component of the Pauli spin operator in the direction of the unit vector $\mathbf{a}$, and $\sigma_{\mathbf{b}} = \sigma \cdot \mathbf{b}$ denote the component of the Pauli spin operator in the direction of the unit vector $\mathbf{b}$. If we "measure" the spin of particle 1 along the direction $\mathbf{a}$ and the spin of particle 2 along the direction $\mathbf{b}$, the results will be correlated, and for the singlet state the correlation is

$$\langle \Psi_0 | \sigma_{\mathbf{a}} \otimes \sigma_{\mathbf{b}} | \Psi_0 \rangle = -\cos\theta_{\mathbf{ab}}, \tag{2}$$

where $\theta_{\mathbf{ab}}$ is the angle between the directions $\mathbf{a}$ and $\mathbf{b}$.

Each spin polarization correlation experiment (A, B) is defined by two macroscopic orientation vectors $\mathbf{A}$ and $\mathbf{B}$ that are some average orientation vectors of the analyzers. An analyzer A is defined by a probability distribution $d\rho_A(\mathbf{a})$, where $\mathbf{a}$ are the microscopic direction vectors,

$$\mathbf{a} \in O_A = \left\{ \mathbf{a} \in S^{(2)}; |1 - \mathbf{a} \cdot \mathbf{A}| \le \varepsilon_A \right\}.$$

Similarly an analyzer B is defined by $d\rho(\mathbf{b})$. The probability $p(A, B)$ that particle 1 is detected by analyzer A and particle 2 (correlated with particle 1) is detected by analyzer B is given by

$$p(A, B) = \eta(A)\,\eta(B) \int_{O_A} \int_{O_B} p_{12}(\mathbf{a}, \mathbf{b})\, d\rho_A(\mathbf{a})\, d\rho(\mathbf{b}), \tag{3}$$

where $p_{12}(\mathbf{a}, \mathbf{b})$ is a probability density function given by quantum mechanics, i.e.,

$$p_{12}(\mathbf{a}, \mathbf{b}) = \frac{1}{2} \sin^2(\theta_{ab}/2)$$

and η's are some factors related to the efficiency of the detectors. Similarly, the predicted correlation function $E(A, B)$ to be compared with the experimental data is given by

$$E(A, B) = \eta(A)\,\eta(B) \int_{O_A} \int_{O_B} -\cos\theta_{\mathbf{ab}}\, d\rho_A(\mathbf{a})\, d\rho(\mathbf{b}). \tag{4}$$

We see that the observable value of the spin projection characterizes only the whole beam of the "particles" which passed through a given analyzer A. Nearly 100 % of the "particles" of this beam would pass by the subsequent identical analyzer A, but we have no prediction concerning any individual "particle" from the beam, and we have no strict spin anti-correlations between the members of each pair.

In the following section, we discuss the various proofs of the Bell's inequalities.

3. Bell's Theorem

For any random experiment, we may find a nonunique mathematical probabilistic model describing it.

Given a probabilistic model there exist, in general, several random experiments which can be described by the model. To obtain consistency of the probabilistic model with the experiment, a particular experimental design and a protocol have to be adopted. This was clearly demonstrated by Bertrand [47] and discussed by us [33, 38].

To each random experiment we associate a random variable X, a probability space S, and a probability density function $f_X(x)$ for all $x \in S$.

If X is a discrete random variable,

$$\sum_x f_X(x) = 1 \qquad \text{and} \qquad P(X = x) = f_X(x).$$

If X is a continuous random variable,

$$\int_S f_X(x)\, dx = 1$$

and

$$P(a \le X \le b) = \int_a^b f_X(x)\, dx, \tag{5}$$

where $P(a \le X \le b)$ is the probability of finding a value of X included between a and b. Note that

$$P(X = x) = 0 \qquad \text{for} \quad \text{all} \qquad x \in S.$$

If in a random experiment we can measure simultaneously the values of k random variables $X_1, \ldots, X_k$, we describe the experiment by a k-dimensional random variable $X = (X_1, \ldots, X_k)$, a common probability space S, and some joint probability density function $f_{X_1 X_2 \ldots X_k}(x_1, \ldots, x_k)$. From the joint probability density function, we can obtain various conditional probabilities and, by integration over $(k-1)$ variables, we obtain k marginal probability density functions $f_{X_i}(x_i)$ describing k different random experiments, each performed to measure only one random variable X_i and neglecting all the others. In this case, we say that $f_{X_i}(x_i)$ were obtained by conditionalization from a unique probability space S. In general, if the random variables X_i are dependent (correlated),

$$f_{X_1 X_2 \ldots X_k}(x_1, \ldots, x_k) \neq f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_k}(x_k). \tag{6}$$

As we found in the preceding section, each spin polarization correlation experiment (A, B) is defined by two macroscopic orientation vectors **A** and **B**, the coincidence probabilities are given by (3), and the correlation functions are given by (4). It is impossible to perform different experiments (A, B) simultaneously on the same couple of particles; therefore, it does not seem possible to use a unique probability space S and to obtain, by conditionalization, the probabilities $p(A, B)$ for all such experiments. This is why it is not so strange that the Bell's inequalities proven using a common probability space do not agree with the predictions of quantum theory.

Let us now analyze the model used by Clauser and Horne [48] to prove their inequalities:

$$p(A, B) = \int_\Lambda p_1(\lambda, A) p_2(\lambda, B)\, d\rho(\lambda), \tag{7}$$

where $p_1(\lambda, A)$ and $p_2(\lambda, B)$ are the probabilities to detect component 1 and component 2, respectively, given the state λ of the composite system.

We see from (7) that the state λ is determined by all the values of strictly correlated spin projections of two components for all possible orientations of analyzers A and B. The analyzers are not perfect, and therefore detection probabilities are introduced. Therefore, it is assumed in the model that even before the detection each component has a well-defined spin projection in all directions. The model uses a single probability space Λ and obtains the predictions on the probabilities $p(A, B)$ measured in different experiments by conditionalization. As we have said, the same assumption was used in all other proofs of the Bell's theorem. An explicit description of states λ by the values of spin projections is also clearly seen in Wigner's proof [49]. As we have said, the experiments (A, B) are mutually exclusive, so there is no justification for using such models.

If we try to prove the Bell's inequalities by comparing only the experimental runs of different experiments, we cannot do it without some additional and questionable assumptions.

Let us simplify the argument we gave in [33]. We want to estimate the value of the spin expectation function $E\,(A, B)$ for the experiment (A, B). We have to perform several runs of length N and find the value of the empirical spin expectation function $r_N(A, B)$ for each run and afterwards to estimate $E\,(A, B)$ by averaging over various runs.

Let us associate with each member of a pair a spin function $s_1(x)$ or $s_2(x)$ taking the values 1 or -1 on the unit sphere $S^{(2)}$ (representing the orientation vectors of various analyzers). We assume also that

$$s_1(x) = -s_2(x) = s(x)$$

for all vectors $x \in S^{(2)}$. We saw in Eq. (3) that the macroscopic directions $\mathbf{A}$ and $\mathbf{B}$ were not sharp; therefore, in each particular run we might have different direction vectors ($\mathbf{a}$, $\mathbf{b}$) representing them. If, for the simplicity, we neglect this possibility, we get

$$r_N(A, B) = -\frac{1}{N} \sum_i s(\mathbf{A}) s_i(\mathbf{B}), \tag{8}$$

where N functions s_i are drawn from some uncountable set of spin functions F_0.

If we consider a particular run of the same length from the experiment (A, C), we get

$$r_N(A, C) = -\frac{1}{N} \sum_i s'_j(\mathbf{A}) s'_j(\mathbf{C}), \tag{9}$$

where N functions s'_j are drawn from the same uncountable set of spin functions F_0.

The probability that we have the same sets of spin functions in both experimental runs is equal to zero. Therefore, in general, we have completely distinct sets of functions in (8) and (9) and we are unable to prove the Bell's theorem by using $r_N(A, B) - r_N(A, C)$. If we use the same sets of spin functions in the runs from the different experiments, then we can replace (9) by (10)

$$r_N(A, C) = -\frac{1}{N} \sum_i s_i(\mathbf{A}) s_i(\mathbf{C}) \tag{10}$$

and we can easily reproduce the Bell's proof, finding his inequalities in the standard form or in the form given for the first time in [19]

$$\left|E(A, B) - E(A, B')\right| + \left|E(A', B') + E(A', B)\right| \leq 2. \tag{11}$$

One may still have some doubts concerning the above argument for the sharp directions of the analyzers. (The samples are not the same, but in the long run everything should average out, etc.) However, if the directions of the analyzers are not sharp, our random experiment is not only a random sampling from some unique population of the spin functions followed by their exact evaluation. In the subquantal description of the experiment (A, B), we have three populations — the population of couples of correlated spin functions, the microscopic directions of analyzer A, and the microscopic directions of analyzer B. A sampling from these three populations produces effective samples of the experimental data, which are sets of couples of numbers ± 1 corresponding to a draw from these populations and the evaluation of the spin functions. Therefore, if we change the experiment into (C, D), the results may not be represented by conditionalization from some unique probability space common for (A, B) and (C, D). The smearing of the polarization directions is important in the impossibility of rigorous proof of the Bell inequalities in this type of subquantal description of the phenomenon.

When the validity of inequality (11) is tested, one should estimate properly all the quantities and include the correct error bars [39].

Let us also note that the act of passage of the ith particle through a given analyzer A depends in a complicated way on its interaction with this analyzer. Therefore, we should not consider a spin function as describing a state of a particle independent of its interaction with A. The spin functions s_i in (8) and (9) resume the interactions of the subsequent particles with the analyzers in a particular experiment. Therefore, if we want to be rigorous, we should replace (8) by (12)

$$r_N(A,B) = -\frac{1}{N}\sum_i s_{i,\mathbf{A}}(\mathbf{a}_i)s_{i,\mathbf{B}}(\mathbf{b}_i), \tag{12}$$

where $\mathbf{a}_i \in O_A$ and $\mathbf{b}_i \in O_B$. If we use formula (12), there is no possibility to prove the Bell's theorem. Using this formula we can always obtain results consistent with Eq. (3). Formula (12) visualizes the contextual character of the observables.

In a trivial but artificial way, a common probability space S can be used in the case where we have four independent experiments described by four independent random variables X_1, X_1', X_2, X_2' and their probability density functions. If all possible values of these variables have absolute value smaller or equal to unity, to prove the Bell's inequalities would be extremely easy. In such a case, the "spin" expectation function $E(X_1, X_2)$ is a product of the expectation values of X_1 and X_2, namely,

$$E(X_1, X_2) = \langle X_1\rangle \langle X_2\rangle$$

and we immediately obtain

$$\begin{aligned} &\left|\langle X_1\rangle \langle X_2\rangle - \langle X_1\rangle \langle X_2'\rangle\right| + \left|\langle X_1'\rangle \langle X_2'\rangle + \langle X_1'\rangle \langle X_2\rangle\right| \\ &\le \left|\langle X_2\rangle - \langle X_2'\rangle\right| + \left|\langle X_2\rangle + \langle X_2'\rangle\right| \le 2, \end{aligned} \tag{13}$$

which is exactly inequality (11).

Of course, if we assume the independence, there are no correlations. The statistical independence is related to the separability of the statistical operator used recently by Krüger in his proofs of the Bell's inequalities in [50].

In the similar way, the quantum correlations are neglected in the cryptographic proof by Herbert [51] reviewed by Ballentine [19]. The source of the singlet state is represented as a generator of two correlated signals. If the two detectors (A, B) are aligned in the same directions, we obtain two messages (the strings of +1 and −1 are identical). If detector B is rotated by an angle θ, it is assumed that the rate of disagreement between the two ($d\theta$) is due only to the change in the orientation of B and does not depend on the orientation of the spatially separated detector A. This assumption leads to the inequality $d(2\theta) \leq 2d(\theta)$, which does not agree with the predictions of quantum mechanics.

Let us note that quantum mechanical correlations are the correlations between the counts of the distant detectors obtained by the coincidence technique and they are never perfect. The messages, strings of the bits, are not send by the source but are only created by the coincidence technique after the results of the measurements for each pair of analyzers (A, B) are recorded. Therefore, in each experiment the rate of disagreement depends on the directions of both macroscopic devices, not only on one of them. Before the measurement there is no message. In Herbert's approach, the rate of disagreement is treated like a measure of the random errors of reading some preexisting incoming message which depends on the rotation of only one of the analyzers from its initial position. The subtle quantum mechanical statistical correlations between counts of A and B are simply ignored and the contextual character of the quantum observables is neglected.

4. Conclusions

The violation of the Bell inequalities requires neither the abandon of the Einstenian separability nor the abandon of the realistic point of view according to which external reality is assumed to exist and to have definite properties. The properties of the reality are, however, not attributive but contextual. Without doubt, in the spin-polarization-correlation experiments, a source is producing the pulses of some real physical field. These pulses are interacting with faraway analyzers and produce the correlated clicks of the detectors. The interference and diffraction of light has been successfully explained by the wave picture of Huygens and Maxwell in the classical physics. The violation of the Bell inequalities forces us to abandon naive realistic models according to which the source is producing a stream of couples of point-like particles flying to the detectors, the couples having well-defined individuality and properties possessed in the attributive way. The subquantal intuitive picture, if it did exist, would have to be of a completely different nature. This subquantal picture is, however, not needed. Quantum theory with its statistical interpretation provides the algorithms allowing one to explain the results of the experiments in the microworld without providing any spatio-temporal description of the physical phenomena involved.

The lack of deterministic predictions for individual measurements and the statistical interpretation of the quantum-state vectors have implications for quantum information. There is no problem with the implementation of quantum cryptography, since transmission of the secret key can be realized successfully with the use of short pulses of polarized light or with Gaussian-modulated coherent states [2] instead of using single photons.

The fact that the quantum state vector is not an attribute of a single quantum system requires more caution in problems related to the implementation of quantum computing devices [1–3]. A more detailed discussion of the contextual character of quantum observables [38,46,52] and its implications for quantum computing will be given in a subsequent paper.

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